Robust Portfolio and Dynamic Disaster Risk

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Abstract

This paper explores the effect of disaster risk on the beliefs and portfolio choices of ambiguity-averse agents. With the introduction of Cressie-Read discrepancies, a time-varying pessimism state variable arises endogenously, generating time-varying disaster risk. In the event of a disaster, agents heighten their pessimism, anticipating subsequent disasters to arrive sooner. Within this framework, we deduce optimal consumption and portfolio choices that are robust to model misspecification. Additionally, our measure of pessimism aids in understanding the stylized facts derived from Vanguard's retail investor survey data, as reported in Giglio et al., 2021.

The seminal work of Hansen and Sargent posits that economic agents, averse to uncertainty, seek robustness by considering a family of models constructed around a benchmark model and optimizing against the worst case within this family. Their approach to robustness uses relative entropy, also known as Kullback-Leibler entropy, to measure the discrepancy between models. This led to a worst-case model for the decision-maker, representing endogenously distorted pessimistic beliefs. Building on this foundation, Maenhout et al., 2021 replace relative entropy with the Cressie and Read, 1984 family of divergences, creating a discrepancy measure that preserves recursivity and homotheticity. This adjustment allows for the emergence of time-varying beliefs in the economy, a phenomenon also uncovered in empirical data. They explore the implications for portfolio choice and asset prices, particularly under continuous asset price movement. In our paper, we further this analysis by incorporating disaster risk and examining the impact of negative jumps in asset prices on ambiguity-averse agents' beliefs and portfolio choices. Our work connects to the literature on time-varying disaster probability, such as studies by Drechsler, 2013 and Wachter, 2013. With the introduction of Cressie-Read discrepancies, our model endogenously generates a time-varying pessimism state variable, leading to time-varying disaster risk, which is negatively related to expected return. This aligns with salient features observed in retail investors' survey data as documented by Giglio et al., 2021.

We summarize our main contributions as follows. We first carefully construct a generalization of Hansen and Sargent's robustness to incorporate disaster into asset prices. Relative entropy is nested as a special case when the Cressie-Read parameter tends to unity. Following Maenhout et al., 2021, agents consider Cressie-Read divergence as a penalty or cost function when minimizing distorted expected utility, comprising an expected integral of suitably scaled and appropriately weighted discounted divergence measures. This approach yields state-dependent belief distortion, except when $\eta = 1$, i.e., in the special case of entropy. The parameter η governs the desire for intertemporal smoothness of belief distortions. We show that for $\eta < 1$, the agent's subjective beliefs are procyclical and become more pessimistic following a disaster. Conversely, $\eta > 1$ leads to countercyclical subjective beliefs. Consequently, subjective disaster risk also varies in response to economic shocks: when $\eta < 1$, it surges after negative jumps and decreases following positive shocks, while the opposite effect occurs for $\eta > 1$.

Second, to further elucidate the underlying mechanisms, we examine a simplified two-stage model where the belief distortion state variable is fixed at the start of each stage. Solving backward, we find that in the second stage, the Cressie-Read investor's beliefs depend only on the fixed state variable set before the second stage. When η is less than one, negative jumps result in a more pessimistic expected return, and when η exceeds one, a more optimistic expected return ensues. In the first stage, the investor anticipates the fact that future utility will depend on the future state of the economy. This anticipation leads to an endogenous hedging demand against future random market conditions and associated belief change. Using this intuition, we then illustrate the consequences of these rich dynamics in a simple partial equilibrium portfolio problem; more specifically, we show that generalized robustness produces intertemporal hedging and, therefore, state- and horizon-dependent portfolios, as well as state- and time-dependent subjective disaster risk. This contrasts with the time-varying disaster literature, where disaster probability is a mean-reverting process, independent of market state. Intuitively, because of time-varying sentiment, disaster belief and investment opportunities are perceived by the investor as time-varying, especially after the disaster happened. This contrasts with the entropy case, where disaster risk and investment opportunities are seen as constant.

Third, we examine the dynamics of pessimism in the simulation. To this end, we reference Giglio et al., 2021, who documented that "higher subjective probabilities of stock market disasters are associated with lower expected stock market returns" in partial equilibrium. When a disaster occurs, the agent becomes more pessimistic, believing that further disasters will soon follow. To test this hypothesis, we estimate the empirical sensitivity of investors' portfolio holding in perceived future disaster risk. Consistent with our theory, we find strong evidence for a negative correlation between subjective probabilities of market disasters and expected returns: disaster probability typically decreases over time but spikes sharply following disasters, reflecting increased pessimism, and decreased expected returns. Furthermore, Monte Carlo simulation reveals that an endogenous time-varying state variable leads to a heavy-tailed distribution of optimal portfolio weight and expected return under subjective belief.

Related Literature

We contribute to several strands of the literature. Our paper builds on the robust control literature such as the works of Hansen and Sargent, 2001, Anderson et al., 2003, and Maenhout, 2004, which imposes an entropy penalty. Maenhout et al., 2021 consider extensions of entropy-based robustness in portfolio choice, demonstrating that robustness with a Cressie-Read penalty produces time-varying beliefs and risk aversion, leading to state- and horizon-dependent portfolios. Our work extends theirs by incorporating disaster risk and generating time-varying risk probability for values of $\eta \neq 1$. Moreover, our model calibration reveals that time-varying pessimism induces a negative correlation between disaster probability and expected return, as shown in survey data.

A large literature in asset pricing studies time-varying disaster risk. For example, Wachter, 2013 demonstrates that time variation in the probability of a consumption disaster drives high stock market volatility and excess return predictability. Drechsler, 2013 constructs an equilibrium model, illustrating that fundamentals and model uncertainty, particularly jump shocks, may explain option prices and the variance premium in both studies, the time-variation in disasters is modeled as an exogenous process: a mean-reverting process in Wachter, 2013 and an autoregressive process in Drechsler, 2013. Different from existing literature, our research illustrates that time-varying disaster risk is endogenous, driven by stochastic sentiment, with the introduction of Cressie-Read discrepancies.

Another related strand of the literature examines the model misspecification concern of jump risk. Liu et al., 2005 explain the skew in index option implied volatilities in equilibrium using investors' uncertainty about rare events, assuming a constant disaster probability and focusing on equilibrium equity prices. Concentrating on the portfolio choice problem,Ait-Sahalia and Matthys, 2019 derive robust consumption and portfolio policies of an investor with recursive preferences, in a model with one risky asset following a Lévy jump-diffusion process. Similarly, Jin et al., 2021 investigate portfolio construction in a multi-asset market setting, considering tail risk and jump ambiguity. While both Ait-Sahalia and Matthys, 2019 and Jin et al., 2021 employ a constant disaster probability, our paper contrasts this approach by introducing subjective time-varying jump risk that arises endogenously, even under a constant jump intensity.

We also contribute to the empirical literature examining investor crash beliefs through survey data. For example, Goetzmann et al., 2016 use survey data to explore the magnitude of crash probabilities reported by individual and institutional investors, finding that individuals' assessment of future crash probability increases following a negative market return. Giglio et al., 2021 further document a negative correlation between subjective probabilities of stock market disasters and expected stock market returns, based on Vanguard's retail investors' survey data. Our research extends the literature by providing a mechanism that explains the observed patterns in expected return and subjective disaster risk, as outlined by Giglio et al., 2021.

Outline of the Paper

The rest of the paper is organized as follows. Section 1 provides the theoretical framework and studies the robust utility in the context of jump risks. Section 2 examines the dynamic portfolio problem. Section 3 presents the quantitative implications of our model for portfolio choices, linking these findings with empirical results documented in the literature. Finally, Section 4 concludes.

1 The Model

1.1 The Cressie-Read Divergence with Jump

1.1.1 Baseline and Subjective Beliefs

Consider a probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{B})$. Here the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmented filtration generated by a standard Brownian motion $B^{\mathbb{B}}$ and a Poisson random measure $N(dt, dz)$, and satisfies the usual hypothesis of right-continuity and completeness. Brownian motion and Poisson random measure are assumed to be independent. The mean measure of N is denoted by $\lambda\nu(dz)dt$, where the jump intensity λ is assumed to be a positive constant and $\nu(dz)$ is the jump size distribution. Denote $\widetilde{N}^{\mathbb{B}}(dt, dz) = N(dt, dz) - \lambda\nu(dz)dt$ as the compensated Poisson random measure.

The measure $\mathbb{B}$ is called the baseline model. The investor worries about model misspecification and entertains a family of alternative models. This family of alternative models is parameterized by a process u and a function θ. Each alternative model $\mathbb{U}$ in this family is defined as follows. Given a bounded process u, introduce Z^D following the dynamics

$$dZ_t^D = -Z_{t-}^D u_t \, dB_t^{\mathbb{B}}, \; 0 \le t \le T, \quad Z_0^D = 1. \tag{1}$$

Given a bounded function $\theta : [0, T] \times \mathbb{R} \to \mathbb{R}$ and a constant $\epsilon > 0$ such that $\theta(t, z) \ge -1 + \epsilon$ for any $(t, z) \in [0, T] \times \mathbb{R}$, introduce Z^J with the dynamics

$$dZ_t^J = Z_{t-}^J \int_{\mathbb{R}} \theta(t, z)\widetilde{N}^{\mathbb{B}}(dt, dz), \; 0 \le t \le T, \quad Z_0^J = 1. \tag{2}$$

Introduce the Radon-Nikodym derivative

$$Z_t = Z_t^J Z_t^D, \quad 0 \le t \le T. \tag{3}$$

Define $\mathbb{U}$ via

$$\left.\frac{d\mathbb{U}}{d\mathbb{B}}\right|_{\mathcal{F}_T} = Z_T.$$[1]

[1]The boundedness assumptions on u and θ ensure $\mathbb{E}^{\mathbb{B}}[Z_T] = 1$. Indeed, Bounded u and θ ensures the Novikov condition $\mathbb{E}^{\mathbb{B}}\left[\exp\left(\frac{1}{2}\int_0^T u_s^2 ds + \int_0^T \int_{\mathbb{R}} \theta^2(s, z)N(ds, dz)\right)\right] < \infty$ in Øksendal and Sulem, 2019 Theorem 1.36, which establishes sufficient conditions such that $\mathbb{E}^{\mathbb{B}}[Z_T] = 1$.

In the alternative model $\mathbb{U}$, $B_t^{\mathbb{B}} = B_t^{\mathbb{U}} - \int_0^t u_s ds$ for a $\mathbb{U}$-Brownian motion $B^{\mathbb{U}}$. Therefore the Brownian motion $B^{\mathbb{B}}$, under the baseline model $\mathbb{B}$, has the drift $-u_t$ under the alternative model $\mathbb{U}$. Meanwhile, the Poisson random measure N has the mean measure $(1+\theta(t,z))\lambda\nu(dz)dt$ under the alternative model $\mathbb{U}$ (see e.g. Øksendal and Sulem, 2019, Theorem 1.33). Hence the jump intensity and the jump size distribution are potentially different under the alternative model. In the special case, where θ is independent of z, the jump intensity becomes $(1+\theta_t)\lambda$ under the alternative model $\mathbb{U}$.

1.1.2 Cressie-Read Divergence

The investor considers alternative models which are further away from the baseline model less likely. Motivated by Maenhout et al., 2021, we introduce the Cressie-Read divergence to measure the discrepancy between the baseline model $\mathbb{B}$ and the alternative model $\mathbb{U}$:

$$R_t^{\mathbb{U}} = \frac{1}{\Phi_t}\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_{s-}dD_{t,s}\Big], \tag{4}$$

where δ is a constant discount rate and $D_{t,s} = \phi(Z_{t,s})$, with $\phi(\cdot)$ as the Cressie-Read divergence function (Cressie and Read, 1984),

$$\phi(z) = \frac{1-\eta+\eta z - z^\eta}{\eta(1-\eta)}, \quad \eta \in \mathbb{R}\backslash\{0,1\}.$$

The function ϕ is convex, satisfies $\phi(1) = 0$, is decreasing when $z \in (0,1)$, and is increasing when $z > 1$. When $\eta = 1$, the Cressie-Read divergence function, defined as the limit $\lim_{\eta\to 1}\phi(z)$, is the KL divergence. When $\eta = 0$, the Cressie-Read divergence, also defined as the limit, corresponds to Burg, 1972 entropy, $\eta = \frac{1}{2}$ describes the Hellinger, 1909 distance.

In (4), Φ and Ψ are two positive processes. Maenhout et al., 2021 show that $\Phi_t = Z_t^{1-\eta}$ is the unique choice so that the resulting Cressie-Read divergence $R^{\mathbb{U}}$ is recursive and time-consistent. The weight process Ψ is chosen later so that the investor's optimization problem remains homothetic.

Lemma 1. *When $\Phi_t = Z_t^{1-\eta}$ and $\mathbb{E}^{\mathbb{B}}\big[\int_0^T e^{-\delta s}\Psi_{s-}^p ds\big] < \infty$ for some $p > 2$, then $R_t^{\mathbb{U}}$ can be decomposed as*

$$R_t^{\mathbb{U}} = R_t^{\mathbb{U},D} + R_t^{\mathbb{U},J},\ 0 \le t \le T, \tag{5}$$

where

$$R_t^{\mathbb{U},D} = \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_{s-}Z_{s-}^{\eta-1}\frac{1}{2}u_s^2 ds\Big], \tag{6}$$

$$R_t^{\mathbb{U},J} = \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_{s-}Z_{s-}^{\eta-1}\lambda\int_{\mathbb{R}}\frac{\big(1+\theta(s,z)\big)^\eta - 1 - \eta\theta(s,z)}{\eta(\eta-1)}\nu(dz)ds\Big]. \tag{7}$$

Moreover, $R^{\mathbb{U}}$ satisfies the following recursive relation

$$R_t^{\mathbb{U}} = \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^{\tilde{t}} e^{-\delta(s-t)}\Psi_{s-}Z_{s-}^{\eta-1}\Big(\frac{1}{2}u_s^2+\lambda\int_{\mathbb{R}}\frac{\big(1+\theta(s,z)\big)^{\eta}-1-\eta\theta(s,z)}{\eta(\eta-1)}\nu(dz)\Big)ds+e^{-\delta(\tilde{t}-t)}R_{\tilde{t}}^{\mathbb{U}}\Big], \quad (8)$$

for any $t, \tilde{t}$ such that $0 \le t \le \tilde{t} \le T$.

When $\eta = 1$, the Cressie-Read divergence is equivalent to the entropy divergence

$$R_t^{\mathbb{U}} = \mathbb{E}_t^{\mathbb{U}}\left[\int_t^T e^{-\delta(s-t)}\Psi_{s-}\Big(\frac{1}{2}u_t^2+\lambda\int_{\mathbb{R}}\Big[\big(1+\theta(t,z)\big)\log\big(1+\theta(t,z)\big)-\theta(t,z)\Big]\nu(dz)\Big)ds\right] \quad (9)$$

which is identical to the relative entropy penalty in Branger and Larsen, 2013, except they only considered jump intensity risk. The jump divergence in (9) is consistent with Liu et al., 2005 or Jin and Zhang, 2012, where they specified distortion θ into intensity and size distortion separately. Our divergence measure is closely related to Ait-Sahalia and Matthys, 2019, where they considered entropy penalty in a relative entropy growth form and specified jump size as beta distribution.

The decomposition (5) split the Cressie-Read divergence into two components: the diffusion component $R^{\mathbb{U},D}$ and the jump component $R^{\mathbb{U},J}$. The diffusion component incorporates a quadratic penalty function $P_{Diffusion}(u) = \frac{1}{2}u^2$ on the diffusion distortion u. The jump component depends on a penalty function P^{η}_{Jump}:

$$P^{\eta}_{Jump}(\theta) = \begin{cases} \frac{(1+\theta)^{\eta}-1-\eta\theta}{\eta(\eta-1)} & \text{if } \eta \neq 1, \\ (1+\theta)\log(1+\theta)-\theta & \text{if } \eta = 1. \end{cases} \quad (10)$$

This penalty function is nonnegative and convex in θ. Figure 1 plots the jump penalty and jump distortion range against θ for different values of η. The left panel of Figure 1 shows that jump penalty $P^{\eta}_{Jump}(\theta)$ is increasing in η when θ is positive and is decreasing in η when θ is negative. Consequently, for a give constraint level κ , the right panel of Figure 1 shows that the interval $\{\theta \ge 0 : P^{\eta}_{Jump}(\theta) \le \kappa\}$ is shrinking in η and $\{-1 < \theta < 0 : P^{\eta}_{Jump}(\theta) \le \kappa\}$ expanding in η. Overall, the investor entertains a larger set of alternative jump models when η is smaller.

When $\eta \neq 1$, the Cressie-Read divergence depends on the state Z. For given diffusion and jump penalty functions $P_{Diffusion}$ and P^{η}_{Jump}, the factor $Z^{\eta-1}$ in front of them in (6) and (7) introduce state dependence into the Cressie-Read divergence. For given u and θ, large $Z^{\eta-1}$ increases the discrepancy. This implies that deviating from $\mathbb{U}$ on states with larger values of $Z^{\eta-1}$ is more costly. Meanwhile, from the dynamics of Z in (1) and (2), we see that the value of Z depends on the historic Brownian shocks and jumps from the Poission random measure. As we will show later, this is the key mechanism to generate endogenous and dynamic beliefs on the diffusion and jump risks. For the portfolio choice problem that

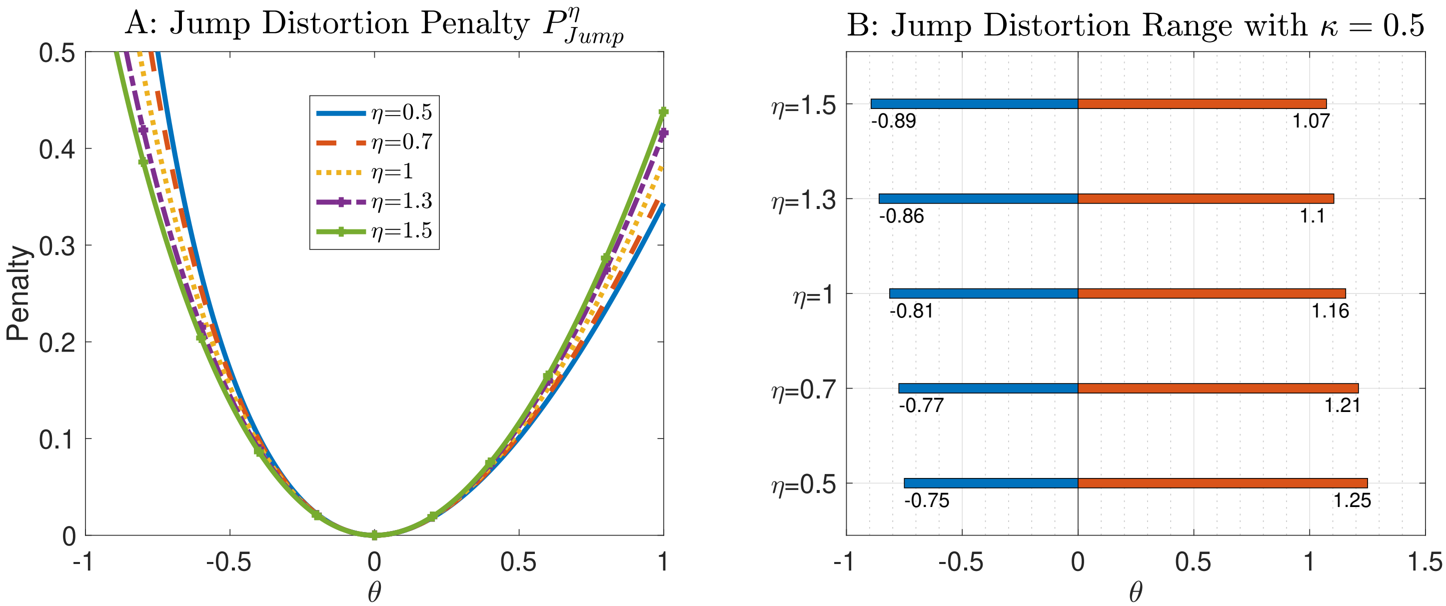

Figure 1: Jump penalty

The left panel plots the jump penalty function P^{η}_{Jump} for different values of η. The right panel presents the interval $\{\theta \geq 0 : P^{\eta}_{Jump} \leq \kappa\}$ and $\{-1 < \theta < 0 : P^{\eta}_{Jump} \leq \kappa\}$ with $\kappa = 0.5$.

the investor considers, Z is the key state variable, which links investor's past experience on diffusion and jump shocks to his belief on future risks. When $\eta = 1$, $Z^{\eta-1} \equiv 1$, hence the Cressie-Read divergence is state independent.

1.2 The Utility Index Process

Given a consumption stream $c = \{c_t, t \in [0, T]\}$, a diffusion distortion u and a jump distortion θ, we define the utility of c under the model $\mathbb{U}$ as

$$\mathcal{U}_t^{c,u,\theta} = \mathbb{E}_t^{\mathbb{U}}\left[\int_t^T e^{-\delta(s-t)}\delta U\left(c_s\right)ds + e^{-\delta(T-t)}\zeta U\left(c_T\right) + \frac{1}{\Theta_D}R_t^{\mathbb{U},D} + \frac{1}{\Theta_J}R_t^{\mathbb{U},J}\right], \tag{11}$$

where U is the utility function for the consumption rate c, ζU is the bequest utility with a positive constant ζ, $R_t^{\mathbb{U},D}$ and $R^{\mathbb{U},J}$ are the diffusion and jump Cressie-Read discrepancy introduced in (6) and (7). The positive constant parameters Θ_D and Θ_J measure investor's preference for robustness of diffusion and jump risk, respectively. When Θ_D (resp. Θ_J) is larger, the investor concerns more about model uncertainty in diffusion (resp. jump) risk.

To evaluate the utility of the consumption stream c among all alternative models, the investor takes a worst case approach. We define the utility index of the consumption stream c as

$$\mathcal{U}_t^c = \inf_{u \in [\bar{u}, \bar{u}], \theta \in (-1, \bar{\theta}]} \mathcal{U}_t^{c,u,\theta}. \tag{12}$$

where $\bar{u} > 0$ and $\bar{\theta} > 0$ are upper bound of process u and function θ respectively.

To illustrate the intuition of the utility index, let us consider an example of a consumption process c following the dynamics

$$\frac{dc_t}{c_{t-}} = \mu_c dt + \sigma_c dB_t^{\mathbb{B}} - \int_{\mathbb{R}} z \widetilde{N}^{\mathbb{B}}(dt, dz), \tag{13}$$

for a constant growth rate μ_c and a constant volatility σ_c. Jumps of the Possion random measure N is assumed to be positive. Hence whenever a jump arrives, consumption drops and we interpret jumps as disasters.

Under alternative measure $\mathbb{U}$, associated with u and θ, the consumption dynamic follows

$$\frac{dc_t}{c_{t-}} = \Big(\mu_c - \sigma_c u_t - \lambda \int_{\mathbb{R}} z\theta(t,z)\nu(dz)\Big) dt + \sigma_c dB_t^{\mathbb{U}} - \int_{\mathbb{R}} z \widetilde{N}^{\mathbb{U}}(dt, dz), \tag{14}$$

where

$$\widetilde{N}^{\mathbb{U}}(dt, dz) = N(dt, dz) - (1 + \theta(t,z))\lambda\nu(dz)dt \tag{15}$$

is a compensated Poisson random measure under $\mathbb{U}$. Consider $u, \theta > 0$, due to the diffusion belief distortion, the expected consumption growth rate decreases by $\sigma_c u$ under $\mathbb{U}$; due to the jump belief distortion, the expected consumption growth rate decreases further by $\lambda \int_{\mathbb{R}} z\theta(t,z)\nu(dz)$. The mean measure of the Poisson random measure N becomes $(1 + \theta(t,z))\lambda\nu(dz)$ under $\mathbb{U}$. In the special case where θ is positive and is independent of z, the jump intensity increases to $(1 + \theta(t))\lambda$ under $\mathbb{U}$ and the jump size distribution remains $\nu(dz)$. Therefore the investor believes the disasters happen more frequently and the expected consumption growth decreases as well.

In formulating the utility index $\mathcal{U}^c$, larger distortion u and θ decrease the consumption growth under the alternative model $\mathbb{U}$, hence the expected utility of consumption (the first two terms on the right-hand side of (11)) decreases. Meanwhile, the Cressie-Read discrepancy (the last two terms on the right-hand side of (11)) increases. The investor chooses the worst case belief distortion u and θ to minimize the expected utility in (12). Given that the Cressie-Read discrepancy $R^{\mathbb{U},D}$ and $R^{\mathbb{U},J}$ depend on Z, Z becomes a state variable for the optimization problem (12), together with the consumption stream c as another state variable.

When $\eta \neq 1$, the utility index $\mathcal{U}^c$ is Markovian in the state variables c and Z, i.e., $\mathcal{U}_t^c = \mathcal{U}\left(t, c_t, Z_t\right)$

for a function $\mathcal{U}$, which satisfies a HJB equation associated to the optimization problem in (12). The following two results summarize the worst case belief distortions u^* and θ^*. The result on u^* is the same as Proposition 1 in Maenhout et al., 2021.

Proposition 1 (diffusion distortion)**.**

1. *The optimal u^* satisfies*
$$u_t^* = \frac{\Theta_D \Gamma_t \left[1 + E_t\right]}{\Psi_t} Z_t^{1-\eta} \tag{16}$$
where $\Gamma_t = \partial_Z \mathcal{U}\left(-Zu^*\right) + \partial_c \mathcal{U}\left(c\sigma_c\right)$ is the instantaneous volatility of the pessimistic utility $\mathcal{U}^c$ and $E_t = \frac{Z_t}{\Gamma_t} \partial_Z \Gamma_t$ is the elasticity of Γ_t with respect to Z_t.

2. *u^* is positive if and only if $\Gamma[1+E]$ is positive. For a fixed and positive $\Gamma[1+E]$, u^* increases as $Z^{1-\eta}$ increases.*

3. *When u^* is positive, positive shocks to $B^{\mathbb{B}}$ decrease Z, hence decrease $Z^{1-\eta}$ when $\eta < 1$, or increase $Z^{1-\eta}$ when $\eta > 1$.*

Proposition 2 (jump distortion)**.**

1. *The optimal θ^* satisfies*
$$\theta^*(t,z) = \left[1 - \frac{(\eta-1)\Theta_J \Lambda_t [1+F_t]}{\Psi_{t-}} Z_{t-}^{1-\eta}\right]^{\frac{1}{\eta-1}} - 1 \tag{17}$$
where $\Lambda_t = \mathcal{U}\big(t, c_{t-}(1-z), Z_{t-}(1+\theta(t,z))\big) - \mathcal{U}\big(t-, c_{t-}, Z_{t-})\big)$ is the utility change due to the negative jump in consumption c and $F_t = \frac{Z_{t-}}{\Lambda_t}\partial_Z \Lambda_t$ is the elasticity of Λ_t with respect to Z_t.

2. *θ^* is positive if and only if $\Lambda[1+F]$ is negative. For a fixed and negative $\Lambda[1+F]$, θ^* increases as $Z^{1-\eta}$ increases.*

3. *When θ^* is positive, Z increases after $N^{\mathbb{B}}$ jumps, hence $Z^{1-\eta}$ increases when $\eta < 1$, or $Z^{1-\eta}$ decreases when $\eta > 1$.*

For entropy discrepancy $\eta = 1$, optimal θ^* becomes
$$\theta^*(t,z) = \exp\left(-\frac{\Theta_J \Lambda_t [1+F_t]}{\Psi_{t-}}\right) - 1 \tag{18}$$
where it's now independent with state variable Z, and one can easily observe that θ^* is positive if and only if $\Lambda[1+F]$ is negative.

Consider a special case where θ in Equation 12 is chosen from the class of processes depending only on time. Under baseline model $\mathbb{B}$,

$$\frac{dc_t}{c_{t-}} = (\mu_c + \lambda \bar{J}_c)dt + \sigma_c dB_t^{\mathbb{B}} - J_t^c dN_t \tag{19}$$

where the disaster in consumption modeled as a Poisson process N_t with rate λ. The change in consumption, should a disaster occur, is J_t^c, a random variable following time-invariant distribution $\nu(dz)$ with mean $\bar{J}_c$. Under alternative measure $\mathbb{U}$, with Poisson rate $\lambda(1+\theta_t)$,

$$\frac{dc_t}{c_{t-}} = \Big(\mu_c - \sigma_c u_t - \lambda \theta_t \bar{J}_c\Big)dt + \sigma_c dB_t^{\mathbb{U}} - \Big(J_t^c dN_t - \lambda(1+\theta_t)\bar{J}_c dt\Big) \tag{20}$$

The optimal θ is independent of z, and follows

$$\theta_t^* = [1 - \frac{(\eta-1)\Theta_J \mathbb{E}^{\nu(dz)}[\Lambda_t][1+F_t]}{\Psi_{t-}} Z_{t-}^{1-\eta}]^{\frac{1}{\eta-1}} - 1 \tag{21}$$

where $\mathbb{E}^{\nu(dz)}[\Lambda_t] = \int_{\mathbb{R}} \big[\mathcal{U}\big(t, c_{t-}(1-J_t^c), Z_{t-}(1+\theta_t)\big) - \mathcal{U}\big(t-, c_{t-}, Z_{t-})\big)\big]\, \nu(dz)$ is the expected marginal utility change due to the negative jump in consumption, and independent of jump size. Similarly, $F_t = \frac{Z_{t-}}{\mathbb{E}^{\nu(dz)}[\Lambda_t]}\mathbb{E}^{\nu(dz)}[\partial_Z \Lambda_t]$ is the expected elasticity of Λ_t with respect to Z_t. In this scenario, the intensity of jumps under $\mathbb{U}$ is $1+\theta_t^*$ times bigger than before but jump size distribution is unaffected, i.e., only jump intensity increases.

2 The Portfolio Choice Problem

We now consider an optimal consumption-investment problem where the investor's utility toward consumption is specified by (12). We focus on the case where the investor concerns about misspecification on the intensity of negative return jumps. We will demonstrate how investor's past experience on negative return jumps shapes his expectation on the likelihood of future disasters and impact on his consumption and investment decisions.

2.1 Model Setup

Consider the market with a constant risk-free rate r, a risky asset with a jump-diffusion dynamics with compound Poisson jumps:

$$\frac{dS_t}{S_{t-}} = (\mu + \lambda \bar{J})dt + \sigma dB_t^{\mathbb{B}} - J_t dN_t \tag{22}$$

where μ is a constant expected return, σ is a constant return volatility, N is a Poisson process with a constant jump intensity λ, $\{J_t\}$ is a sequence of i.i.d. random variables with distribution ν on $(0,1)$, and $\bar{J}$ is the mean jump size.

Given a bounded process u and a function $\theta : [0,T] \to \mathbb{R}$, the risky asset follows the following dynamics under the alternative model $\mathbb{U}$:

$$\frac{dS_t}{S_{t-}} = \left(\mu - \sigma u_t - \lambda \bar{J}\theta_t\right) dt + \sigma dB_t^{\mathbb{U}} - (J_t dN_t - \lambda(1+\theta_t)\bar{J}dt), \tag{23}$$

where $dB_t^{\mathbb{U}} = u_t dt + dB_t^{\mathbb{B}}$ and N has the time-dependent intensity $\lambda(1+\theta_t)$. Therefore the agent's subjective expected return is $\mu - \sigma u_t - \lambda \bar{J}\theta_t$. When u and θ are both positive, the subjective expected return under $\mathbb{U}$ is reduced compared with the objective expected return under $\mathbb{B}$. In contrast to Ait-Sahalia and Matthys, 2019, our model does not feature the trade off between a higher frequency of jumps and a higher jump risk premium. Larger θ increases jump frequency under $\mathbb{U}$ and reduces asset's expected return as well.

Given a portfolio weight π and a consumption rate c, investor's wealth process follows the dynamics

$$dW_t = \Big[rW_{t-} + W_{t-}\pi_t(\mu + \lambda\bar{J} - r) - c_t\Big]dt + W_{t-}\pi_t\sigma dB_t^{\mathbb{B}} - W_{t-}\pi_t J_t dN_t. \tag{24}$$

The investor chooses his optimal strategy (π, c) to maximize the utility index of consumption

$$V_t = \sup_{(\pi,c)\text{ admissible}} \mathcal{U}_t^c, \tag{25}$$

subject to the wealth dynamics (24). The utility index $\mathcal{U}_t^c$ is given in (12). The strategy (π, c) is admissible if it is predictable with respect to investor's filtration $\mathcal{F}$ and $\pi_t \leq 1/J_{\max}$ for any t, where $J_{\max}$ is a positive constant, strictly less than 1, such that the jump size distribution ν has a compact support inside $[0, J_{\max}]$. Therefore, $\pi_t J_t < 1$ for any t so that $W > 0$ at all time.

The utility for intertemporal and bequest consumption is $U(c_s) = \frac{c^{1-\gamma}}{1-\gamma}$ with the relative risk aversion $0 < \gamma \neq 1$. To maintain homotheticity with respect to investor's wealth, we follow Maenhout, 2004 and choose $\Psi_t = (1-\gamma)V_t$ in (6) and (7).[2]

Combining the form of $R^{\mathbb{U}}$ in Lemma 1, $\mathcal{U}_t^c$ in (12), and the choice of Ψ yields the value function (25) as

$$V_t = \sup_{\pi,c} \inf_{u,\theta} \mathbb{E}_t^{\mathbb{U}} \Big[\int_t^T e^{-\delta(s-t)} \Big(\delta U\left(c_s\right) + (1-\gamma) V_{s-} Z_{s-}^{\eta-1} P(u_s, \theta_s) \Big) ds + e^{-\delta(T-t)} \epsilon U\left(c_T\right) \Big], \qquad (26)$$

where

$$P(u_s, \theta_s) = \frac{1}{2\Theta_D} u_s^2 + \frac{\lambda}{\Theta_J} \frac{\left(1+\theta_s\right)^\eta - 1 - \eta\theta_s}{\eta(\eta-1)}. \qquad (27)$$

is obtained from Cressie-Read divergence. In (26), V_t depends on future values V_s, $s \in [t,T]$, problem (26) is an optimization problem for a recursive utility. The choice of Ψ maintains the homotheticity with respect to investor's wealth, the value function can be decomposed as

$$V_t = V_t(W_t, Z_t) = \frac{W_t^{1-\gamma}}{1-\gamma} f(t, Z_t), \qquad (28)$$

for some function f.

2.2 A Two-stage Example

We now illustrate the intuition of the problem (25) using a two-stage example. In the first stage, $t \in [0,1)$; in the second stage, $t \geq 1$. In stage 1, the investor does not update the value of Z, i.e., $Z_t \equiv Z_0$ for $t \in [0,1)$. The investor chooses the belief distortion (u_0, θ_0) at time 0 and keep it constant until time 1. At time 1, the investor updates Z to Z_1, using the shocks experienced before time 1. The investor updates the belief distortion to (u_1, θ_1), and chooses the portfolio and consumption strategy for the second stage. In stage 2, $Z_t \equiv Z_1$ for $t \geq 1$. Therefore the investor keeps the belief distortion (u_1, θ_1) and the consumption-investment strategy constant in the second stage.

We solve the two-stage problem by backward induction. For the second stage, agent's optimal consumption-investment problem is

$$V_t = \inf_{u_1,\theta_1} \sup_{\pi,c} \mathbb{E}_t^{\mathbb{U}} \Big[\int_t^\infty e^{-\delta(s-t)} \Big(\delta U\left(c_s\right) + (1-\gamma) V_{s-} Z_1^{\eta-1} P(u_1, \theta_1) \Big) ds \Big], \quad t \geq 1, \qquad (29)$$

[2]Because $\mathcal{U}_t^c$ depends on V_s for $s \in [t,T]$, problem (25) becomes an optimization problem for a recursive utility; see also Maenhout et al., 2021

subject to wealth dynamic (24).

Because the problem is homothetic in wealth and $Z_1^{\eta-1}$ is a constant in (29). The investor's optimal belief distortion and consumption-investment strategy are constants and can be obtained by solving an algebraic equation (A.14). The dependence of optimal belief distortion and consumption-investment strategy on $Z_1^{\eta-1}$ is summarized in the following result.

Proposition 3. *When $Z_1^{\eta-1}$ increases, the agent's optimal portfolio weight π_1 increases, diffusion distortion u_1 and jump distortion θ_1 decrease.*

When $u_t \equiv u_0$ and $\theta_t = \theta_0$ for any $t \in [0, 1)$, $Z_1 = \exp\big(-\frac{1}{2}u_0^2 - \lambda\theta_0 - u_0 B_1^{\mathbb{B}} + \ln(1+\theta_0)N_1\big)$. Suppose that $u_0 > 0$ and $\theta_0 > 0$, we now examine the impact of shocks on optimal portfolio and belief distortions. If $\eta < 1$, positive shocks to $B_1^{\mathbb{B}}$ increase $Z_1^{\eta-1}$, leading the agent to increase her portfolio weight π_1^* in risky asset, and decrease her belief distortion u_1^* and θ_1^*, according to Proporition 3. After negative shocks to $B_1^{\mathbb{B}}$ or negative jumps (i.e. disasters) in stock returns, $Z_1^{\eta-1}$ decreases. Consequently, agent reduces π_1^* while increasing u_1^* and θ_1^*. When $\eta > 1$, the investor behaves in an opposite way.

We focus on the impact of negative jumps in returns. In the case $\eta < 1$, as previously discussed, two belief distortions increase after negative jumps, generating a more pessimistic expected return $\mu - \sigma u_1 - \lambda \bar{J}\theta_1$. Meanwhile, the investor's subjective disaster intensity $\lambda(1+\theta_1)$ increases, hence the investor expects more frequent future disasters in his worst-case belief. As the investor becomes more pessimistic, he reduces his portfolio weight in the risky asset. In this case, the pessimism is countercyclical with the realized return. When $\eta > 1$, after negative jumps, investor's subjective disaster intensity decreases, and the expected return increases. The investor becomes less pessimistic and increases his optimal portfolio weight π_1^*. Therefore pessimism is procyclical with the realized return. Negative return shocks from $B_1^{\mathbb{B}}$ share a similar impact with negative jumps. Positive return shocks lead to an opposite effect. Table 1 summarizes these results.

Table 1: Responses to Exogenous Shocks

	Z_1	u_1^*	θ_1^*	π_1^*	Sentiment	Subjective Disaster Intensity
			$\eta < 1$			
$\Delta B^{\mathbb{B}} > 0$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$	less pessimistic	
$\Delta B^{\mathbb{B}} < 0$ or disaster	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$	more pessimistic	$\uparrow$
			$\eta > 1$			
$\Delta B^{\mathbb{B}} > 0$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	more pessimistic	
$\Delta B^{\mathbb{B}} < 0$ or disaster	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$	less pessimistic	$\downarrow$

For the first stage, the agent's value function is

$$V_t = \inf_{u_0,\theta_0} \sup_{\pi,c} \mathbb{E}_t^{\mathbb{U}} \Big[\int_t^1 e^{-\delta(s-t)} \Big(\delta U\left(c_s\right) + (1-\gamma) V_{s-} Z_0^{\eta-1} P(u_0,\theta_0) \Big) ds + e^{-\delta(1-t)} V_1 \Big], \quad 0 \le t \le 1, \tag{30}$$

where $Z_0 = 1$. Because V_1 equals to $\frac{W_1^{1-\gamma}}{1-\gamma} f(Z_1)$ and depends on Z_1, intertemporal hedging demand emerges in the first stage, against market volatility, disaster probability, and changing belief distortion at time 1.

2.2.1 Numerical Illustrations

We now use a numerical example to illustrate the dependence of optimal portfolio choice and worst-case belief distortion in the second stage on the state variable Z_1. We assume a constant return jump size for illustration purpose.

Table 2: Parameter Values

Parameter	Variable	Value
r	Interest Rate	0.03
δ	Discount Rate	0.03
μ	Expected Stock Return	0.1
σ	Stock Volatility	0.2
λ	Constant Jump Intensity	0.1
J	Stock Jump Size	0.2
γ	Risk Aversion	6
Θ_D	Preference Parameter for Diffusion Ambiguity	1
Θ_J	Preference Parameter for Jump Ambiguity	1

For illustration, we introduce a monotone transformation $x = -\log Z_1$, so that when $u_0, \theta_0 > 0$, x increases (decreases) after positive (negative) shocks. Figure 2 presents two cases with $\eta < 1$ and $\eta > 1$, respectively. Panel A in Figure 2 presents the case $\eta = 0.5$. Positive shocks increases x, the investor becomes less pessimistic, his belief distortion u_1^* and θ_1^* both reduce, expected return increases, and portfolio holding π_1^* increases. Panel B presents the case $\eta = 1.5$. Positive shocks increases x, the investor becomes more pessimistic, his belief distortions increase, both expected return and portfolio holding decline. These results are consistent with Table 1.

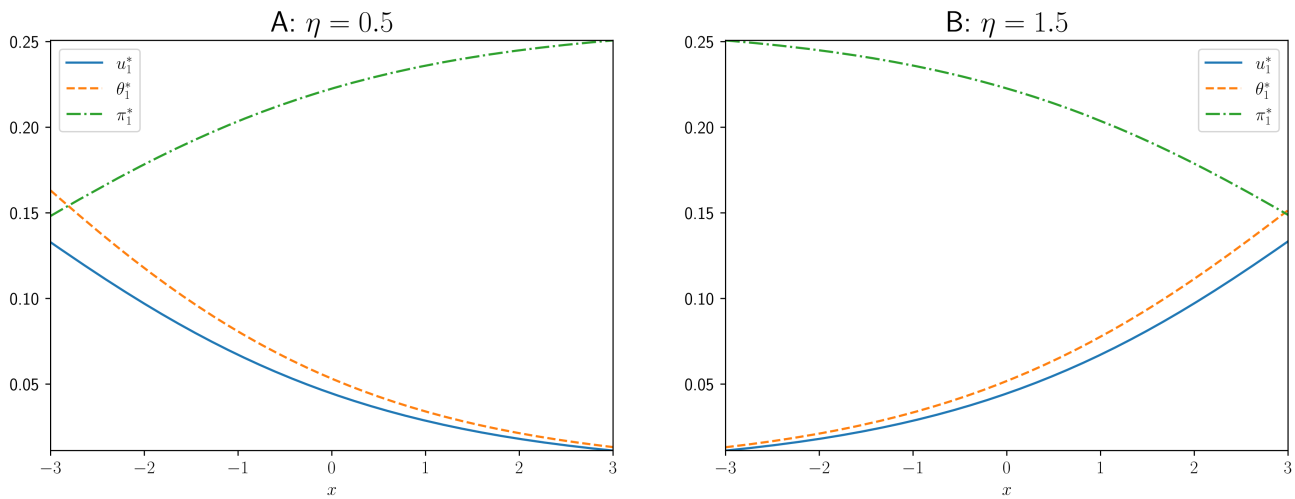

Figure 2: Optimal portfolio and belief distortions

This figure plots worst-case belief distortions (u_1^*, θ_1^*) and the optimal portfolio weight π_1^* in the second stage. The parameters used are summarized in Table 2.

2.3 Dynamic Optimal Consumption and Portfolio Choice

Having built up intuition from the two-stage example, we examine the fully dynamic problem (26) and identify the optimal consumption, portfolio choice, and the worst-case belief. We introduce the following variable x as the state variable of the problem (26):

$$x_t = -\log Z_t. \tag{31}$$

We derive from (1) and (2) that x follows the dynamics

$$dx_t = \left(\frac{1}{2}u_t^2 + \lambda\theta_t\right)dt + u_t dB_t^{\mathbb{B}} - \ln\left(1+\theta_t\right)dN_t. \tag{32}$$

When u and θ are positive, jumps and negative Brownian shocks decrease x, meanwhile positive Brownian shocks increase x. We call x the *sentiment variable* and use it as the state variable for problem (26).

We introduce two constants $\underline{x}$ and $\overline{x}$ satisfying $\underline{x} < 0 < \bar{x}$. We fix $\underline{x}$ (resp. $\overline{x}$) to be sufficiently small (resp. large), so that $x > \overline{x}$ (resp. $x < \overline{x}$) correspond to Z close to 0 (resp. large Z). We regard it as unreasonably underweight (resp. overweight) in the alternative model $\mathbb{U}$. Define a stopping time

$$\tau = \inf\left\{t \geq 0 : x_t \leq \underline{x} \text{ or } x_t \geq \bar{x}\right\}. \tag{33}$$

When $t > \tau$, we freeze Z_τ at e^{-x_τ} and set the Cressie-Read discrepancy in (11) as

$$\frac{1}{\Theta_D} R_t^{\mathbb{U},D} + \frac{1}{\Theta_J} R_t^{\mathbb{U},J} = \mathbb{E}_t^{\mathbb{U}} \left[\int_t^T e^{-\delta(s-t)} \Psi_s e^{(1-\eta)x_\tau} \left(\frac{1}{2\Theta_D} u_s^2 + \frac{\lambda}{\Theta_J} \frac{\left(1+\theta_s\right)^\eta - 1 - \eta\theta_s}{\eta(\eta-1)} \right) ds \right]. \quad (34)$$

As we will see in Theorem 4, this specification helps to pin down boundary conditions for the HJB equation.

The choice $\Psi = (1-\gamma)V$ maintaines the homothetic property for problem (26). Take W and x as the state variables, the value function V in (26) has the following decomposition

$$V_t = \frac{W_t^{1-\gamma}}{1-\gamma} e^{f(t,x_t)}. \quad (35)$$

The following result characterizes the function f and identifies agent's optimal consumption investment strategies and the worst-case belief distortion.

Proposition 4. *Suppose that f is a classical solution of the following HJB equation*

$$\begin{aligned} 0 = \sup_{\pi,\tilde{c}} \inf_{\theta,u} \Bigg\{ & \frac{\partial_t f}{1-\gamma} + \partial_x f \left[\pi\sigma u - \frac{1}{2(1-\gamma)} u^2 + \frac{\lambda}{1-\gamma}\theta \right] + \frac{(\partial_x f)^2 + \partial_{xx} f}{2(1-\gamma)} u^2 \\ & + \frac{\lambda(1+\theta)}{1-\gamma} \left\{ \mathbb{E}_\nu \left[(1-\pi J)^{1-\gamma} \right] e^{f(x-\ln(1+\theta)) - f(x)} - 1 \right\} + \frac{\delta}{1-\gamma} \tilde{c}^{1-\gamma} e^{-f} \\ & + r + \pi \left(\mu - r - \sigma u + \lambda \bar{J} \right) - \tilde{c} - \frac{1}{2}\gamma\pi^2\sigma^2 - \frac{\delta}{1-\gamma} + e^{(1-\eta)x} P(u,\theta) \Bigg\} \end{aligned} \quad (36)$$

for $(t,x) \in [0,T) \times (\underline{x}, \bar{x})$, and the boundary conditions

$$f(t,x) = f_{\underline{x}}^{fr}(t),\ x \le \underline{x}, \quad f(t,x) = f_{\bar{x}}^{fr}(t),\ x \ge \bar{x}, \quad \textit{and} \quad f(T,x) = \log\zeta.$$

The function P in (36) *is given in* (27).

Then, the optimal belief distortions u^* and θ^*, the optimal investment strategy π^*, and the optimal *consumption-wealth ratio* $\tilde{c}^* = \frac{c^*}{W}$ *satisfy the following first-order conditions*

$$u^* = \frac{(1-\gamma)(1-\partial_x f)}{(\partial_x f)^2 + \partial_{xx} f - \partial_x f + \frac{1-\gamma}{\Theta_D} e^{(1-\eta)x}} \sigma \pi^*, \tag{37}$$

$$0 = \frac{1}{\Theta_J} e^{(1-\eta)x} \frac{(1+\theta^*)^{\eta-1} - 1}{\eta - 1} + \frac{1-\partial_x f}{1-\gamma} \left\{ e^{f(t,x-\ln(1+\theta^*))-f(t,x)} \mathbb{E}_\nu \left[(1-\pi^* J)^{1-\gamma} \right] - 1 \right\}, \tag{38}$$

$$\mu - r = \gamma^{\mathit{eff}} \pi^* \sigma^2 - \lambda \bar{J} + \lambda (1+\theta^*) e^{f(t,x-\ln(1+\theta^*))-f(t,x)} \mathbb{E}_\nu \left[(1-\pi^* J)^{-\gamma} J \right], \tag{39}$$

$$\tilde{c}^* = \delta^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma} f(x)}, \tag{40}$$

where $\gamma^{\mathit{eff}} = \gamma + \frac{(1-\gamma)(1-\partial_x f)^2}{(\partial_x f)^2 + \partial_{xx} f - \partial_x f + \frac{1-\gamma}{\Theta_D} e^{(1-\eta)x}}$.

When $x \le \underline{x}$ *or* $x \ge \bar{x}$, f_x^{fr} *satisfies the following ODE*

$$\begin{aligned} 0 = & \frac{\partial_t f_x^{fr}}{1-\gamma} + \frac{\delta}{1-\gamma} \delta^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma} f_x^{fr}} + \frac{\lambda(1+\theta^{fr})}{1-\gamma} \left\{ \mathbb{E}_\nu [\left(1-\pi^{fr} J\right)^{1-\gamma}] - 1 \right\} \\ & + r + \pi^{fr} \left(\mu - r - \sigma u^{fr} + \lambda \bar{J} \right) - \tilde{c}^{fr} - \frac{1}{2} \gamma {\pi^{fr}}^2 \sigma^2 - \frac{\delta}{1-\gamma} + e^{(1-\eta)x} P(u^{fr}, \theta^{fr}), \end{aligned} \tag{41}$$

with boundary terminal conditions $f_x(T) = \log \zeta$. *In* (41), u^{fr}, θ^{fr} *and* π^{fr} *satisfy*

$$u^{fr} = \Theta_D e^{(\eta-1)x} \sigma \pi^{fr}, \tag{42}$$

$$\theta^{fr} = \left[\left\{ \mathbb{E}_\nu \left[\left(1-\pi^{fr} J\right)^{1-\gamma} \right] - 1 \right\} \frac{1-\eta}{1-\gamma} \Theta_J e^{(\eta-1)x} + 1 \right]^{\frac{1}{\eta-1}} - 1, \tag{43}$$

$$\mu - r = \left(\gamma + \Theta_D e^{(\eta-1)x} \right) \sigma^2 \pi^{fr} - \lambda \bar{J} + \lambda (1+\theta^{fr}) \mathbb{E}_\nu \left[\left(1-\pi^{fr} J\right)^{-\gamma} J \right]. \tag{44}$$

Equation (39) implies that the risk premium $\mu - r$ can be attributed into hedging demand from diffusion risk and from jump risk. The first term on the right-hand side of (39), $\gamma^{\text{eff}} \pi^* \sigma^2$, is the risk premium associated to the diffusion risk. We interpret γ^{eff} as the effective risk aversion, which is belief- and state-dependent. The rest terms on the right-hand side of (39) constitute the jump risk premium. $\lambda (1+\theta^*)$ is subjective jump intensity and $e^{f(x-\ln(1+\theta^*))-f(x)} \mathbb{E}_\nu \left[(1-\pi^* J)^{-\gamma} J \right]$ is the ratio of marginal values of investment after and before a return jump. As the investor anticipates future changes in the sentiment variable, a Merton-type intertemporal hedging demand emerges. In (38), the optimal belief distortion in jump θ^* is determined by equating the marginal cost $\frac{1}{\Theta_J} e^{(1-\eta)x} \frac{(1+\theta^*)^{\eta-1}-1}{\eta-1}$ with the marginal gain $\frac{\partial_x f - 1}{1-\gamma} \left\{ e^{f(x-\ln(1+\theta^*))-f(x)} \mathbb{E}_\nu \left[(1-\pi^* J)^{1-\gamma} \right] - 1 \right\}$.

When $\eta = 1$, the optimal solutions in (37) - (39) are state independent and satisfy:[3]

$$u^* = \Theta_D \sigma \pi^* \tag{45}$$

$$0 = \frac{1}{\Theta_J} \log(1+\theta^*) + \frac{1}{1-\gamma} \left\{ \mathbb{E}_\nu \left[(1-\pi^* J)^{1-\gamma} \right] - 1 \right\} \tag{46}$$

$$\mu - r = (\gamma + \Theta_D)\pi^* \sigma^2 - \lambda \bar{J} + \lambda \left(1 + \theta_t\right) \mathbb{E}_\nu \left[(1-\pi^* J)^{-\gamma} J \right]. \tag{47}$$

This is consistent with Liu et al., 2005, who consider both jump intensity and size uncertainty using the entropy discrepancy.

2.4 Numerical Results

In this section, we present our model implications using numerical solutions of the HJB equation (36).

2.4.1 State Dependence

Investor's worst-case belief and optimal portfolio are state-dependent. To illustrate this, we consider a short horizon problem with 1 year time horizon.

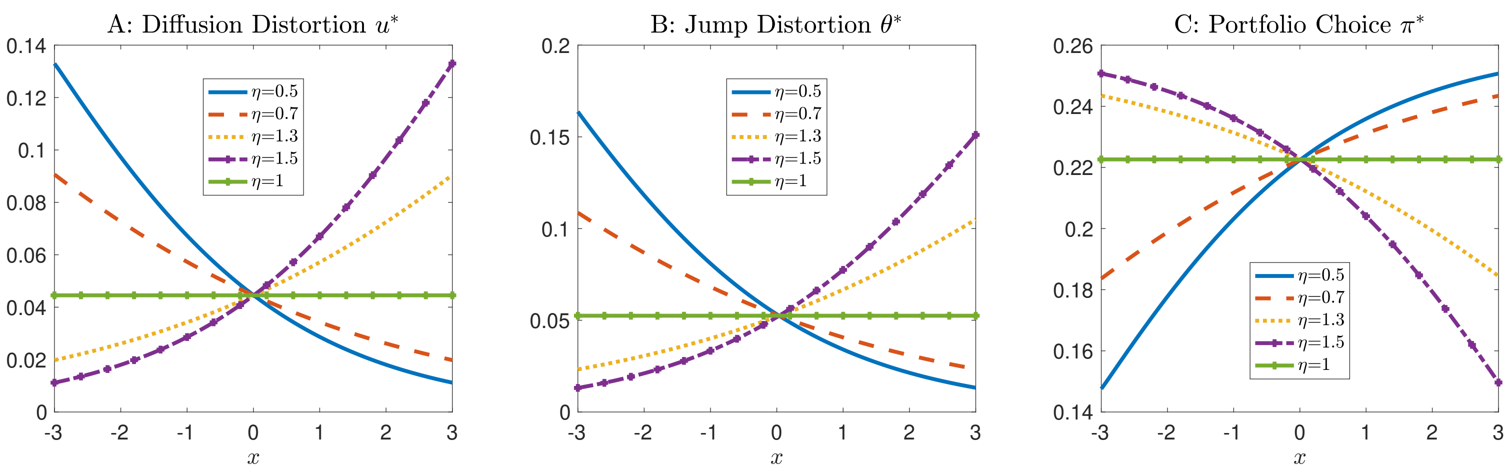

Figure 3: State dependence for different η

This figure presents the worst-case belief distortions and the optimal portfolio weight at time zero. The time horizon is T=1 year and other parameters are summarized in Table 2.

Figure 3 presents the diffusion distortion u^*, jump intensity distortion θ^*, and portfolio choice π^* for

[3] The case of $\eta = 1$ is discussed in Appendix C

different values of η. When $\eta = 1$, the discrepancy measure in (8) is the entropy case. In this case, Figure 3 shows that portfolio allocation and belief distortions are state independent.

When $\eta < 1$, as x increases, sentiment improves, both u and θ decrease, the investor becomes less pessimistic, and increases risky asset weight the portfolio. When $\eta > 1$, as x decreases, sentiment deteriorates, both u and θ increase, the investor becomes more pessimistic, and invests less in the risky asset. Moreover, Figure 3 shows the further η is away from 1, the more state dependence of the belief distortions and portfolio strategy are.

We now change the jump intensity or jump size to study the impact of jump on the state-dependence.

Figure 4 presents the optimal (u^*, θ^*, π^*), compensations for diffusion risk and jump risk across various values of jump intensity λ. When λ increases, the jumps come more frequently, and new arrivals become less surprising. As a result, θ^* becomes smaller as λ increases, as Figure 4 Panel B illustrates. Nevertheless, larger λ still increases $\lambda \bar{J} \theta^*$, which reduces the expected return under the worst-case belief $\mathbb{U}$ (see (23)). Therefore, the portfolio weight π^* decreases, as Figure 4 Panel C shows. Panel D shows that the state-dependent term $\frac{(1-\gamma)(1-\partial_x f)}{(\partial_x f)^2 + \partial_{xx} f - \partial_x f + \frac{1-\gamma}{\Theta_D} e^{(1-\eta)x}}$ on the right-hand side of (37) remains roughly the same as λ increases. Therefore (37) shows that u^* decreases with π^* as λ increases. Even though the reduction of u^* increases the expected return under the worst-case belief $\mathbb{U}$, the reduction from jump component, $\lambda \bar{J} \theta^*$, still dominates, so that the expected return $\mu - \sigma u^* - \lambda \bar{J} \theta^*$ decreases as λ increases. Given risk premium $\mu - r = 0.07$, higher jump intensity induces larger jump risk, requiring higher compensation for jump risk, while the compensation for diffusion risk decreases. This trade-off between diffusion risk and jump risk is shown in Panel E and F in Figure 4.

Figure 5 shows the impact of jump size on belief, portfolio allocation, and risk premium decomposition. Arrival of jumps with larger jump size is more surprising, therefore θ^* increases with the jump size. This increases $\lambda \bar{J} \theta^*$ hence reduces the expected return under the worst-case belief $\mathbb{U}$ and the portfolio weight π^*. Meanwhile, as Figure 5 shows, the state-dependent term on the right-hand side of (37) remains roughly the same as λ increases. Therefore, u^* decreases with π^* as the jump size increases. Meanwhile, larger jump size demands higher compensation for jump risk, hence lower compensation for diffusion risk.

2.4.2 Time-to-horizon Effect

To examine the intertemporal hedging, we increase the time horizon to T=100 years and present the optimal distortions and portfolio weight at $x = 0$ for different η, jump intensity and size.

Figure 6 shows myopic behavior for the entropy case $(\eta = 1)$. When $\eta \neq 1$, the diffusion distortion u^*, jump intensity distortion θ^* and portfolio choice π^* depend on time-to-horizon.

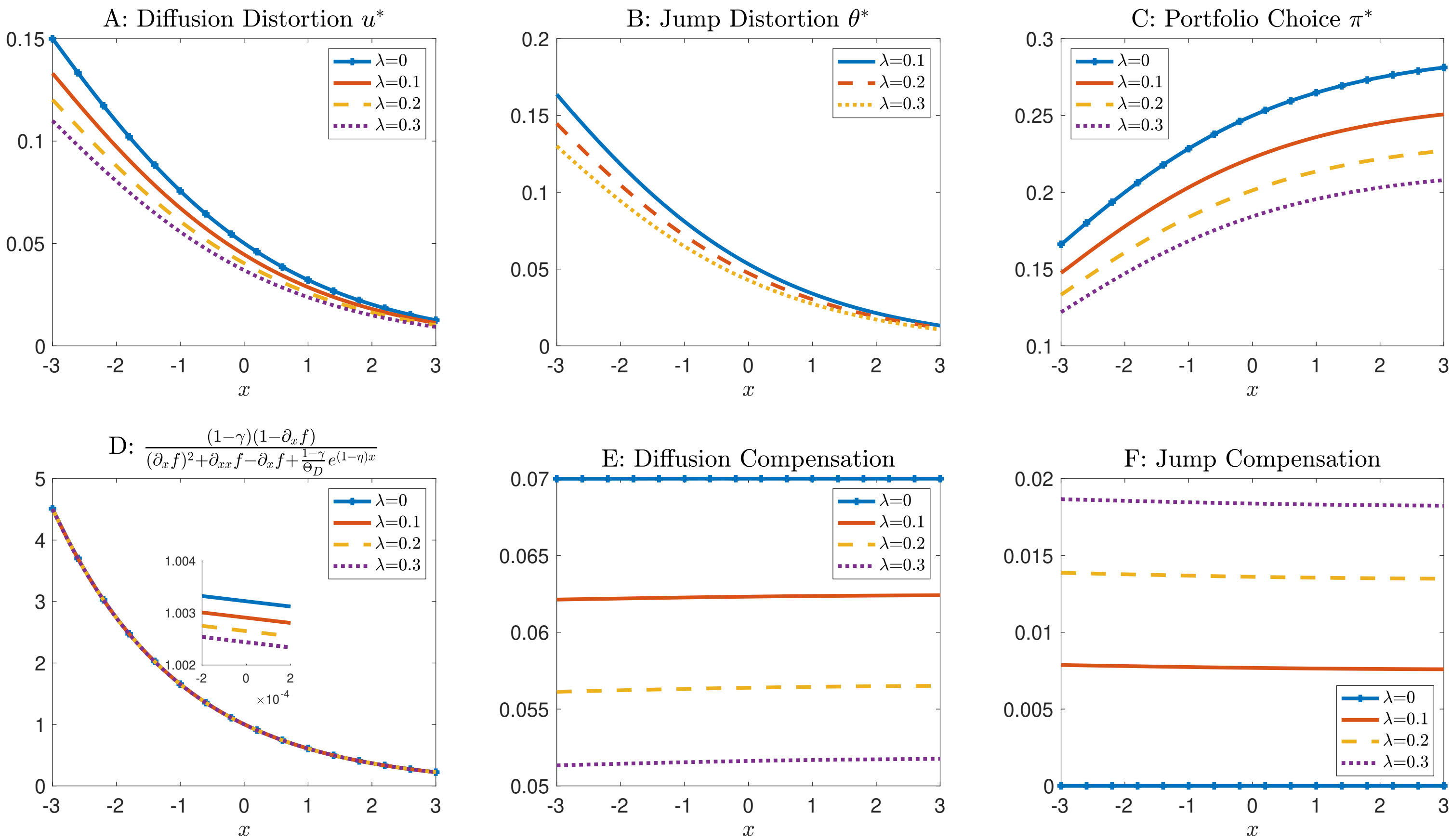

Figure 4: State dependence for different λ

This figure plots optimal distortions, portfolios, and jump intensity change at time zero. The time horizon is T=1 year and other parameters are summarized in Table 2.

Following the same intuition of the two-stage example, when $\eta < 1$, pessimism is countercyclical, a disaster in return is associated with a deterioration in sentiment, which deepens pessimism and decreases the subjective expected return on the risky asset. In other words, when $\eta < 1$, realized returns and subjective expected returns are positively correlated, which leads to negative intertemporal hedging demands that grow with the time-to-horizon, shown in Figure 6 in the case $\eta = 0.5$. When $\eta > 1$, pessimism is procyclical, a disaster in return leads to reduction in pessimism and increase in the subjective expected return. Therefore, a negative correlation emerges between realized returns and the subjective returns. This explains the positive intertemporal hedging demand for the case $\eta = 1.5$ in Figure 6 Panel C. This hedging demand also grows as time-to-horizon increases.

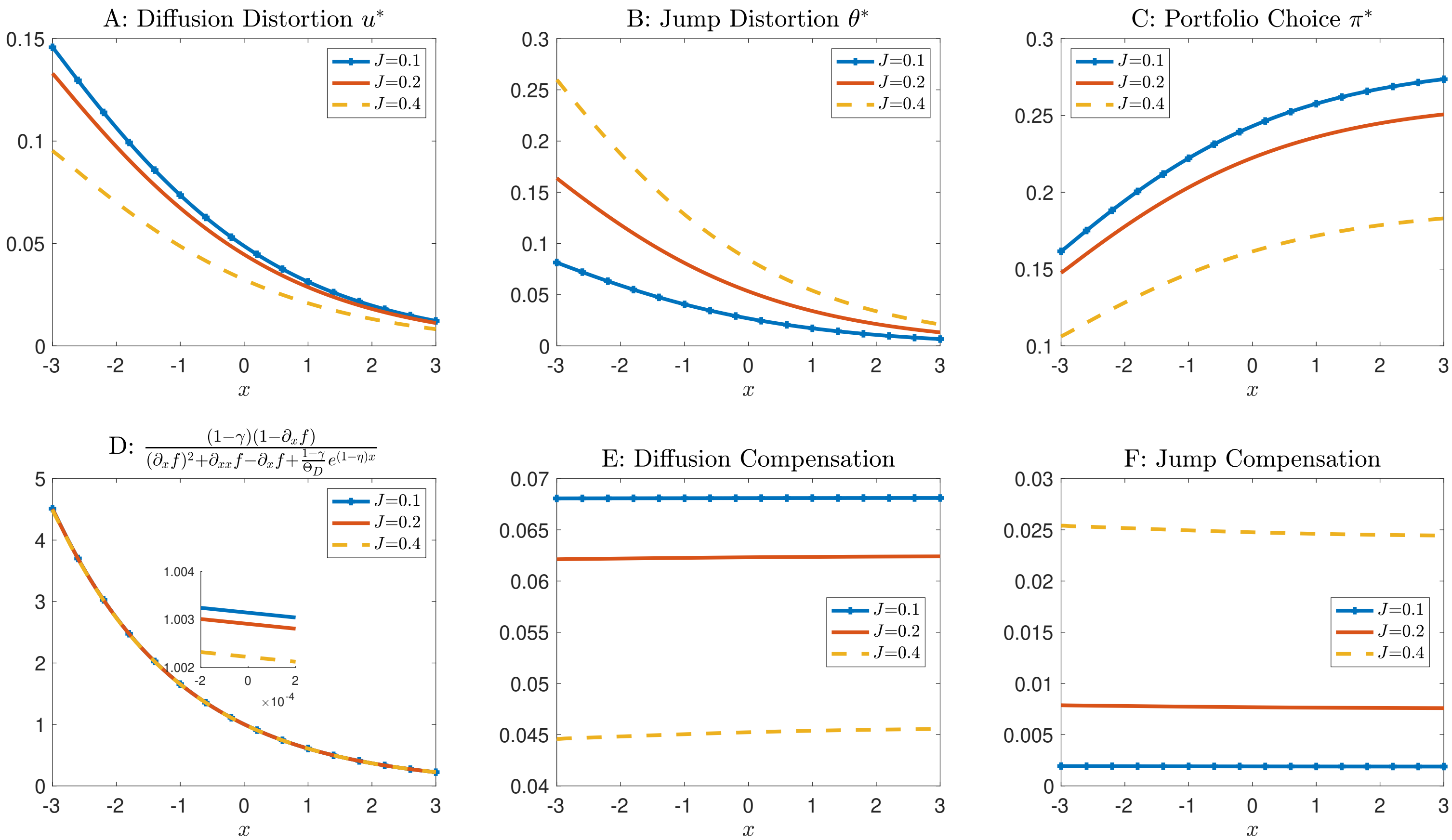

Figure 5: State dependence for different J

This figure plots optimal distortions and portfolios at time zero. The time horizon is T=1 year and other parameters are summarized in Table 2.

2.4.3 Dynamic response to jumps

We now examine the effect of jumps on the dynamics of state variable, the belief distortion, and portfolio choice via a simulation exercise. We simulate the dynamics of x following (32) for 30 years. We choose $\theta_J = 20$ so that the investor has strong concern in jump intensity ambiguity. This choice highlights the jump effect in our model.

Figure 7 presents a sample path. The worst-case jump belief distortion θ^* is positive. Therefore, in absent of jumps, the state variable x drifts upward, due to its positive drift $\frac{1}{2}|u_t^*|^2 + \lambda\theta_t^*$ in (32). Upon arrival of Poisson jumps, represented by red vertical lines in Figure 7, x jumps down. We choose $\eta = 0.7$ for our simulations. Therefore the worst-case jump belief distortion θ^* decreases with x, as Panel B of

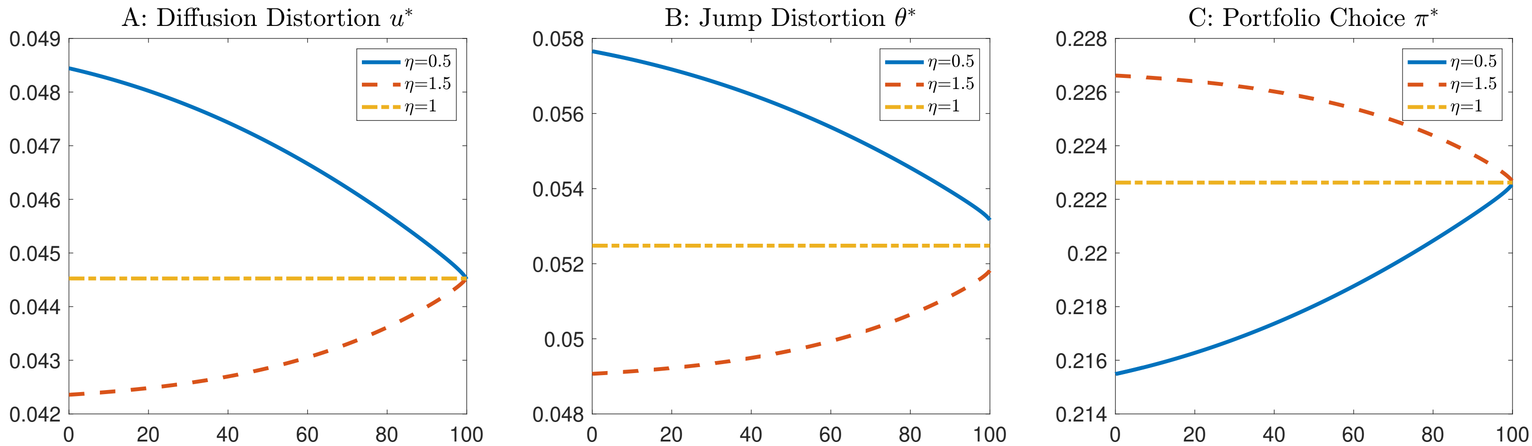

Figure 6: Optimal distortion and portfolio

This figure plots optimal distortions and portfolios at $x = 0$. The time horizon is T=100 years and other parameters are summarized in Table 2.

Figure 3 indicates. Consequently, the jump size $\ln(1 + \theta_t^*)$ is larger when x is lower. Because θ^* is positive, the subjective jump intensity $\lambda(1+\theta^*)$ is always larger than the objective jump intensity $\lambda = 0.1$. As Panel B of Figure 7 shows, in absent of jumps, the subjective jump intensity declines. This is because the state variable x increases and θ^* decreases with the state variable. Upon arrival of Poisson jumps, due to the downward jump of the state variable, the subjective jump intensity jumps up. Therefore, there is a self-exciting pattern in investor's belief in the likelihood of future disasters: when the investor observes a return disaster, he anticipates a higher probability of disasters in the future; if he experiences a period without disasters, he also anticipates a lower probability of disasters in the future. In contrast to Hawkes processes, whose objective jump intensity increases after arrivals of jumps, the effect described before is purely subjective, the objective jump intensity remains a constant $\lambda = 0.1$ regardless of the jump arrivals. Nevertheless, changes in agent's subject belief on jumps impact his subjective expected return and his portfolio choice. Panel C of Figure 7 shows that investor's subjective expected return drifts upward in absent of disasters, but jumps downward upon arrival of disasters, because the investor anticipates a higher likelihood of disasters in the future. Consequently, the investor's portfolio weight increases following his improved sentiment, but jumps downward after arrivial of each disaster, as Panel D of Figure 7 indicates. In contrast, the model with entropy divergence, i.e., $\eta = 1$, displays state and time-independent subjective jump intensity and portfolio choice, see the dotted dash line in Figure 7.

Compared with Bayesian learning, the state variable drives investor's belief and portfolio choice in our model. If there is a disaster, the state variable, expected return, and portfolio weight change significantly, no matter how many disaster have happened before. When an investor learns an unknown jump intensity

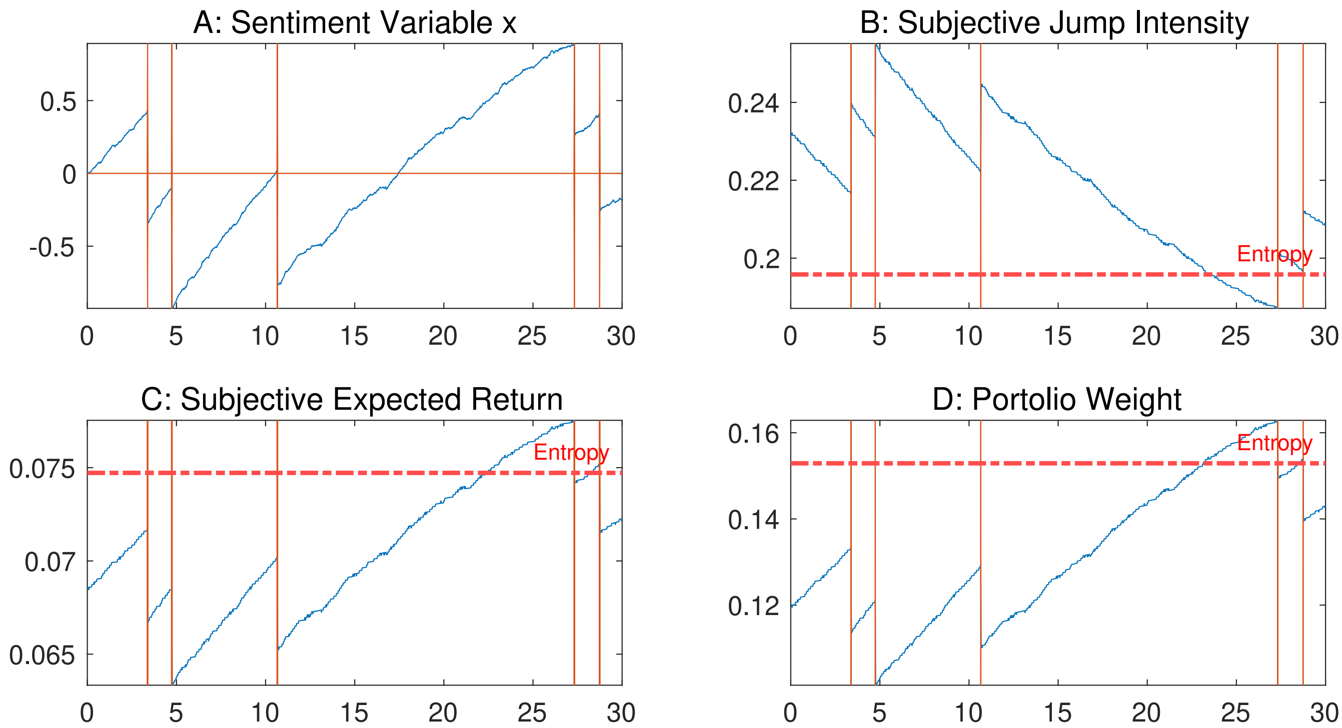

Figure 7: A Simulation Path When $\eta < 1$

The figure plots a simulated path of state variable, subjective disaster probability, expected return, and portfolio choice from 0 to 30 years. The parameters used are summarized in Table 2 with $\theta_D = 1$, $\theta_J = 20$, $\eta = 0.7$, $\underline{x} = -3$, and $\bar{x} = 3$.

parameters, his posterior belief on the intensity parameters becomes more accurate as he observes more disasters, hence future disasters have decreasing impact on investor's posterior belief and less impact on investor's portfolio choice.

Figure 8 presents the distribution of the state variable, subjective jump intensity, subjective expected return, and portfolio choice. All of them presents a heavy-tailed nature, resulting from higher subjective jump intensity after disasters.

Figure 9 presents a sample path for the case with $\eta > 1$. The state variable jumps downward after each return disaster. However, in contrast to the $\eta < 1$ case, the subjective jump intensity drops after each disaster. The investor anticipates smaller likelihood of future disasters given that one just happened. As a result, both the subjective expected return and the portfolio weight jump upward after each disaster.

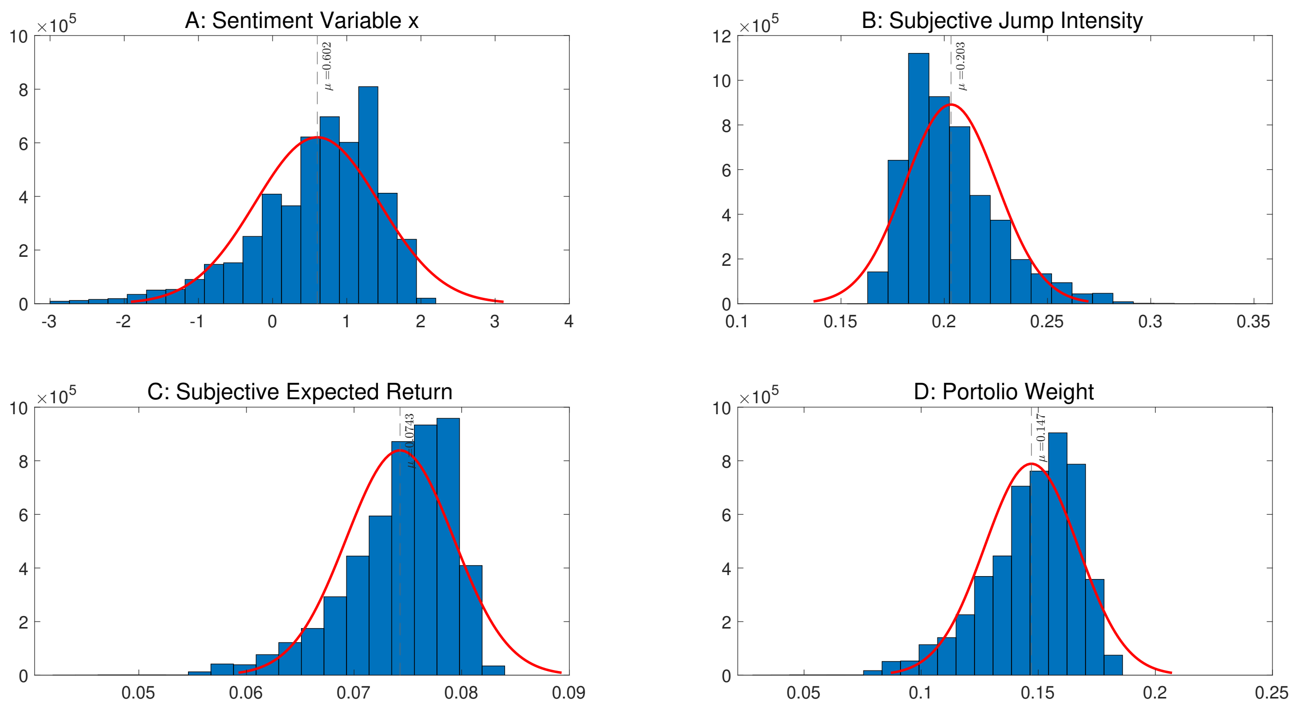

Figure 8: Simulation Distribution Based on Our Model

This figure plots the histograms of sentiment variable, expected return, jump probability, and portfolio weight between years 10 and 20, with T=30 among 10^4 simulated path. The parameters used are summarized in Table 2 with $\theta_D = 1$, $\theta_J = 20$ and $\eta = 0.7$.

3 Connection with survey evidence

Giglio et al., 2021 examines a series of surveys administered to Vanguard retail investors. These surveys elicit investors' belief on future economy states such as GDP growth and stock market expected returns. The authors match beliefs of respondents with their portfolio composition, trading activity, and login behavior. The study reveals that investors' belief influence their portfolio allocations and trading decisions, demonstrates significant individual heterogeneity in beliefs, and identifies the relationships between expected returns and expected cash flow growth, as well as the one year subjective probability of rare disasters.

In particular, Giglio et al., 2021 documents that investors who believe a higher probability of disaster (1-year stock returns of less than -30%) has a lower subjective expected return and equity share in their portfolios. In this section, we examine the same linear regressions (subjective expected returns and risky asset weights on the subjective disaster probability) as in Giglio et al., 2021 using our model simulated

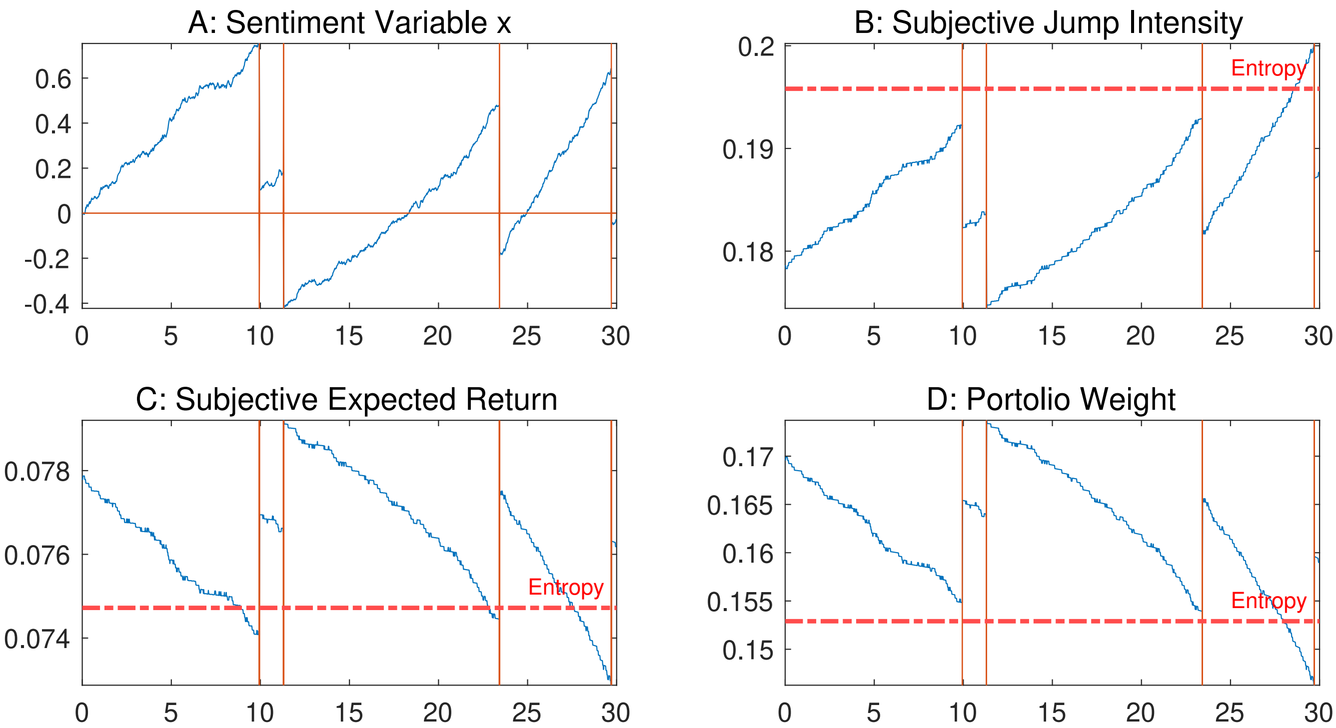

Figure 9: A Simulation Path When $\eta > 1$

The figure plots a simulated path of state variable, subjective disaster probability, expected return, and portfolio choice from 0 to 30 years. The parameters used are summarized in Table 2 with $\theta_D = 1$, $\theta_J = 20$, $\eta = 1.3$, $\underline{x} = -3$, and $\bar{x} = 3$.

data and compare the regression results with the survey evidence documented by Giglio et al., 2021.

We simulate our model using parameters in Table 3. We choose the equity premium $\mu - r = 7.39\%$ and the return volatility $\sigma = 0.1598$ from Campbell, 2017. The return jump size is set to be 0.3, mapping to the empirical specifications in Giglio et al., 2021. The jump intensity λ measures the objective disaster probability in a year. Therefore we map the model jump intensity to the empirical disaster probability, and use jump intensity and disaster probability interchangeably in this section. We choose $\lambda = 0.1$ so that the resulting mean of subjective disaster probability roughly agrees with the empirical mean from surveys. We set the Cressie-Read parameter to be $\eta = 0.8$ so that disasters introduce more pessimistic belief and there is a countercyclic portfolio weight. We choose Θ_J larger than θ_D so that the investor has more model ambiguity concern on disaster probability because disasters are rare events in historic data. The investment horizon T is set to be 30 years mapping to the average investment horizon of a retail investor before retirement.

Figure 10 illustrates distributions of the state variable, subjective disaster probability, subjective

Table 3: Parameter Values

Parameter	Variable	Value
r	Interest Rate	0.02
δ	Discount Rate	0.02
μ	Expected Stock Return	0.0939
σ	Stock Volatility	0.1598
λ	Stock Jump Intensity	0.1
J	Stock Jump Size	0.3
γ	Risk Aversion	2
η	Cressie-Read Parameter	0.8
Θ_D	Preference Parameter for Diffusion	1
Θ_J	Preference Parameter for Jump	5
T	Time Horizon	30

We choose $\underline{x} = -3$ and $\bar{x} = 3$ and simulate 10^4 paths of the state variables starting from $x_0 = 0$. To mitigate the dependence on the initial value, we use the simulated panel data at $t = 5$ year for the model-based regressions.

expected return, and portfolio weight among simulated panel data at $t = 5$ year. Compared to the objective belief, the subjective belief exhibits a higher disaster probability due to positive jump belief distortion θ^*. The average subjective disaster probability is 20.4% consistent with the empirical mean in Giglio et al., 2021. The average subjective expected return is approximately 5.4%, and the average portfolio weight is about 34.1%. This is consistent with the high portfolio weight found by Giglio et al., 2021, even in the absence of a high risk aversion level.

Figure 11 presents the relationship between the subjective expected return with the state variable. Because η is less than 1, the subjective expected return increases with the state variable. Meanwhile, due to positive jump belief distortion θ^*, disasters reduce the level of state variable. Hence investor's subjective expected return jumps downward after each disaster. Focusing on time-series, rather than cross-sectional, of disasters in simulated paths, our simulation shows that the model-generated subjective expected return decreases on average by 4.75% compared with its pre-disaster level when disasters occur; meanwhile, the portfolio weight decreases by 8.6% compared with pre-disaster level.

Giglio et al., 2021 estimate the sensitivity of the subjective expected return to the subjective disaster probability with the following regression:

$$\text{Subjective Expected Return}_i = \alpha + \beta \times \text{Subjective Disaster Probability}_i + \epsilon_i, \tag{48}$$

where each i corresponds to each survey respondent. Meanwhile, the risky asset weight is influenced by a variety of factors. Giglio et al., 2021 examine the relationship between the risky asset weight and the subjective disaster probability, the subjective expected return, and the volatility of expected return via the

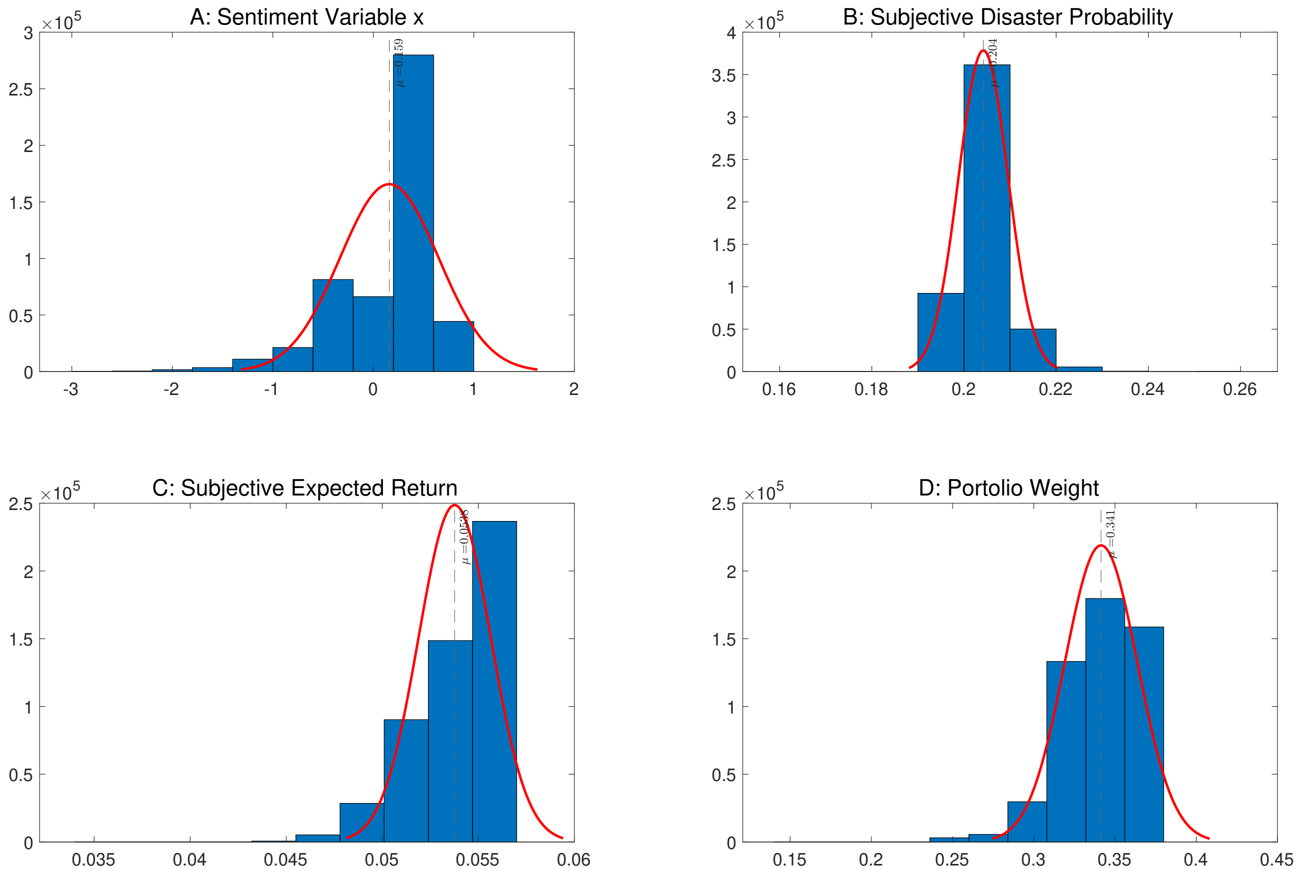

Figure 10: Simulation Distribution

This figure plots the histograms of sentiment variable, expected return, jump probability, and portfolio weight in the fifth year, with T=30. The parameters used are summarized in Table 3.

regression:

$$\begin{aligned} \text{Risky Asset Weight}_i = \alpha + \beta_1 \times \text{Subjective Disaster Probability}_i \\ + \beta_2 \times \text{Subjective Expected Return}_i \\ + \beta_3 \times \text{Standard Deviation Expected Return}_i + \epsilon_i. \end{aligned} \tag{49}$$

We simulate 10^4 paths of the state variable and average the subjective disaster probability, the subjective expected return, and risky asset weight within the tenth year within the simulation i to construct the variables Subjective Disaster Probability$_i$, Subjective Expected Return$_i$, and Risky Asset Weight$_i$,

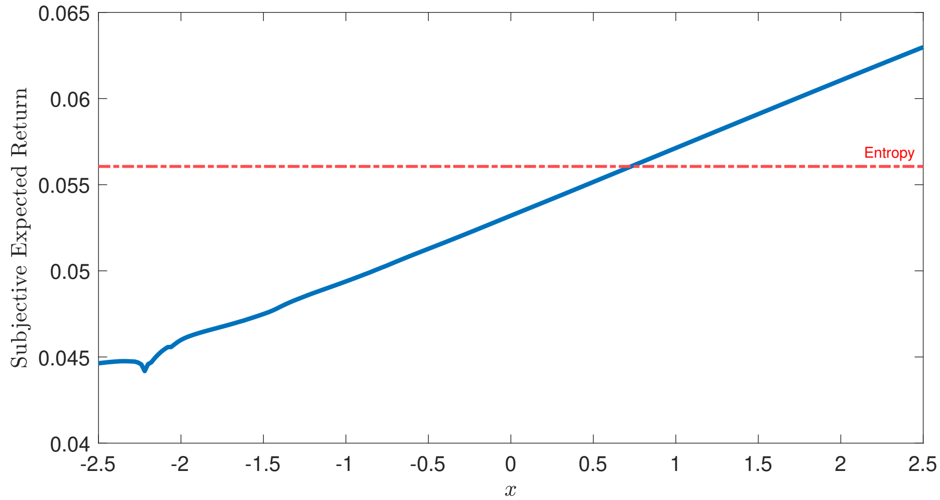

Figure 11: Expected Return under Subjective Belief

This figure plots subjective expected return against sentiment variable x, with fixed t at the fifth year and parameters summarized in Table 3.

respectively. Using the simulated data, we examine the regressions (48) and (49).

Table 4 summarizes the model-generated results. Our model generates similar impact of subjective disaster probability on the subjective expected return and portfolio weight as Giglio et al., 2021. Column 2 in Table 4 shows that one percent increase in the subjective disaster probability is associated with a 0.352 percent decrease in the subjective expected return, compared to 0.146 percent decrease in Giglio et al., 2021. Meanwhile, Column 4 shows that an percent increase in the subjective disaster probability decreases the risky asset weight by 4.149 percent in our model. In our model, this reduction is due to two factors: the reduction in the subjective expected return and the increase in investor's implicit risk aversion. When controlling for expected return and volatility of return, one percent increase in the subjective disaster probability reduces the weight of risky asset by 0.574 percent, comparing to 0.116 percent decrease in Giglio et al., 2021. Even though the subjective expected return and subjective disaster probability are highly correlated in our model, the coefficient of disaster probability remains significant after introducing the subjective expected return as a control.

In our simulation, a one-standard-deviation increase in the subjective disaster probability is associated with a 2.19 percentage point lower risky asset weight. This consists with the result in Giglio et al., 2021 that "A one-standard-deviation increase in the perceived stock market disaster probability is associated with about a one percentage point lower equity share."

Table 4

	Model Simulation			Giglio et al. 2021	
	Expected Return (%)	Risky Asset Share (%)		Expected Return (%)	Risky Asset Share (%)
	(1)	(2)	(3)	(4)	(5)
Subjective Disaster Probability (%)	-0.352***	-4.149***	-0.574**	-0.146**	-0.116**
	(0.000)	(0.001)	(0.045)	(0.013)	(0.027)
Expected Return (%)			10.208**		0.737**
			(0.026)		(0.040)
Standard Deviation Expected Return (%)			-0.003***		-0.036*
			(0.002)		(0.052)
Constant	0.126***	1.189***	-0.090**	–	–
	(0.000)	(0.000)	(0.016)		
R^2	1	0.999	1	0.722	–
N	10,000	10,000	10,000	43,492	44,235

The regression is based on the simulated data in the fifth year, with T = 30 years. The subjective disaster probability and expected return are averaged within the fifth year for each path to exclude individual effects. The standard deviation of expected return is calculated by annualized volatility for each path over the fifth year. The model parameters for simulation are summarized in Table 3.

$^{*}p : 0.1, ^{**}p : 0.05, ^{***}p : 0.01.$

4 Conclusion

The paper analyzes the portfolio planning problem in a jump-diffusion model when there is model uncertainty. The investor is ambiguity averse and relies on a robust control approach, following Maenhout et al., 2021. Different from their paper, we incorporate uncertainty about both jump risk as well as diffusion risk. After disaster strikes, the agent becomes more pessimistic and thinks further disaster will arrive soon. However, if disaster still does not strike yet, the agent becomes gradually less pessimistic.

We find that diffusion and jump misspecification will impact the optimal portfolio decision. Furthermore, under the robust measure, disaster probability increases after disasters happen and decrease the expected return. Moreover, endogenous time-varying state variable in our model leads to heavy-tailed distributions of optimal portfolio weight and expected return under subjective belief.

Our study emphasizes the importance of jump tail behavior in optimal portfolio formation. It provides a mechanism to rationalize the negative correlation between investors' subjective expected return and rare disaster risk, documented in survey data (Giglio et al., 2021).

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A Proofs

A.1 Proof of Lemma 1

When u and θ are both bounded and that θ is bounded away from -1, we claim that $\mathbb{E}^{\mathbb{B}}[Z_{t,s}^q]$ is bounded uniformly in $s \in [t, T]$ for any $q \in \mathbb{R}$. To see this, we obtain from (3) that

$$\begin{aligned} Z_{t,s}^q = \exp\Big\{ \int_t^s \Big[-\frac{q}{2}u_v^2 + \int \big[q\ln(1+\theta(v,z)) - q\theta(v,z)\big]\lambda\nu(dz)\Big]dv - \int_s^t qu_v dB_v^{\mathbb{B}} \\ + \int_s^t \int q\ln(1+\theta(v,z))\widetilde{N}^{\mathbb{B}}(dv,dz)\Big\}. \end{aligned}$$

Because u and θ are both bounded and θ is bounded away from -1, the claim follows from taking $\mathbb{B}$-expectation on both sides.

Itô's formula for processes with jumps (see e.g. Protter, 2005, Chapter II, Section 7 yields

$$\begin{aligned} dD_{t,s} = &-\frac{Z_{t,s-} - Z_{t,s-}^\eta}{1-\eta}u_s dB_s^{\mathbb{B}} - \int_{\mathbb{R}} \Big[\frac{\theta(s,z)}{\eta-1}Z_{t,s-} - \frac{(1+\theta(s,z))^\eta - 1}{\eta(\eta-1)}Z_{t,s-}^\eta\Big]\widetilde{N}^{\mathbb{B}}(ds,dz) \\ &+ \frac{1}{2}Z_{t,s-}^\eta u_s^2 ds + \int_{\mathbb{R}} Z_{t,s-}^\eta \lambda \frac{\big(1+\theta(s,z)\big)^\eta - 1 - \eta\theta(s,z)}{\eta(\eta-1)}\nu(dz)ds. \end{aligned} \tag{A.1}$$

It follows from Hölder's inequality that

$$\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}(Z_{t,s-} - Z_{t,s-}^\eta)^2\Psi_{s-}^2|u_s|^2 ds\Big] \le C\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}(Z_{t,s-} - Z_{t,s-}^\eta)^{2q}ds\Big]^{\frac{1}{q}}\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_{s-}^{2p}ds\Big]^{\frac{1}{p}},$$

where $C = \max|u|$ and $1/p + 1/q = 1$. Because $\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_{s-}^{2p}ds\Big] < \infty$ by assumption and $\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}(Z_{t,s-} - Z_{t,s-}^\eta)^{2q}ds\Big] < \infty$, the process $\int_t^\cdot e^{-\delta(s-t)}(Z_{t,s-} - Z_{t,s-}^\eta)u_s dB_s^{\mathbb{B}}$ is a martingale under $\mathbb{B}$. Meanwhile,

$$\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_{s-}|\theta(s,z)|Z_{t,s-}\lambda\nu(dz)ds\Big] \le C\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}Z_{t,s-}^{2q}ds\Big]^{\frac{1}{2q}}\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_{s-}^{2p}ds\Big]^{\frac{1}{2p}},$$

where the constant C depends on $\max|\theta|, \delta$ and T. Both expectations on the right-hand sides are finite. It then follows from Remark 6.4 c) in Cinlar, 2011, Chapter VI that $\int_t^\cdot \Psi_{s-}\int_{\mathbb{R}} \frac{\theta(s,z)}{\eta-1}Z_{t,s-}\widetilde{N}^{\mathbb{B}}(ds,dz)$ is a martingale under $\mathbb{B}$. The same statement holds for $\int_t^\cdot \Psi_{s-}\int_{\mathbb{R}} \frac{(1+\theta(s,z))^\eta - 1}{\eta(\eta-1)}Z_{t,s-}^\eta\widetilde{N}^{\mathbb{B}}(ds,dz)$ as well.

Choose $\Phi_t = Z_t^{1-\eta}$ and use the martingale properties obtained above, we obtain

$$\begin{aligned} R_t^{\mathbb{U}} &= \frac{1}{\Phi_t}\mathbb{E}_t^{\mathbb{B}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_{s-}dD_{t,s}\Big] \\ &= \mathbb{E}_t^{\mathbb{U}}\Big[\int_t^T e^{-\delta(s-t)}\Psi_{s-}Z_{s-}^{\eta-1}\Big(\frac{1}{2}u_s^2 + \lambda\int_{\mathbb{R}}\frac{\big(1+\theta(s,z)\big)^\eta - 1 - \eta\theta(s,z)}{\eta(\eta-1)}\nu(dz)\Big)ds\Big]. \end{aligned} \tag{A.2}$$

Define $R_t^{\mathbb{U},D}$ and $R_t^{\mathbb{U},J}$ as in (6) and (7), respectively. We have $R_t^{\mathbb{U}} = R_t^{\mathbb{U},D} + R_t^{\mathbb{U},J}$.

A.2 Proofs of Propositions 1 and 2

The dynamics of Z_t follows

$$\frac{dZ_t}{Z_{t-}} = -u_t dB_t^{\mathbb{B}} + \int_{\mathbb{R}}\theta(t,z)\Big(N(dt,dz) - \lambda\nu(dz)dt\Big). \tag{A.3}$$

By Itô formula for jump-diffusion, the utility index $\mathcal{U}_t^c$ follows the dynamic

$$\begin{aligned} d\mathcal{U}_t^c =& \partial_t\mathcal{U}dt - \partial_Z\mathcal{U}\cdot Z_{t-}\left(u_t dB^{\mathbb{B}} + \lambda\int_{\mathbb{R}}\theta(t,z)\nu(dz)dt\right) + \partial_c\mathcal{U}\cdot c_{t-}\left(\sigma^c dB^{\mathbb{B}} + \big(\mu_c + \lambda\int_{\mathbb{R}} z\nu(dz)\big)dt\right) \\ &+ \frac{1}{2}\partial_{ZZ}^2\mathcal{U}\cdot(u_t Z_{t-})^2 dt + \frac{1}{2}\partial_{cc}^2\mathcal{U}\cdot(\sigma_c c_{t-})^2 dt - \partial_{Zc}^2\mathcal{U}\cdot(u_t Z_{t-})(\sigma_c c_{t-})dt \\ &+ \int_{\mathbb{R}}\Big\{\mathcal{U}\big(t, c_{t-}(1-z), Z_{t-}(1+\theta(t,z))\big) - \mathcal{U}\big(t, c_{t-}, Z_{t-}\big)\Big\}N(dt,dz). \end{aligned} \tag{A.4}$$

We can solve the the optimal distortions for utility index (12) via martingale approach, as

$$e^{-\delta t}Z_t\mathcal{U}_t^c + \int_0^t e^{-\delta s}Z_s\left\{\delta U(c_s) + \Psi_{s-}Z_{s-}^{\eta-1}\Big(\frac{1}{2\Theta_D}u_s^2 + \frac{\lambda}{\Theta_J}\int_{\mathbb{R}}\frac{\big(1+\theta(s,z)\big)^\eta - 1 - \eta\theta(s,z)}{\eta(\eta-1)}\nu(dz)\Big)\right\}ds \tag{A.5}$$

is $\mathbb{B}$-martingale under optimal u and θ. Applying Itô formula to pervious equation, we have

$$\begin{aligned} &-\delta e^{-\delta t}Z_t\mathcal{U}_t^c + e^{-\delta t}d\left(Z_t\mathcal{U}_t^c\right) \\ &\quad + e^{-\delta t}Z_t\left\{\delta U(c_t) + \Psi_{s-}Z_{s-}^{\eta-1}\Big(\frac{1}{2\Theta_D}u_s^2 + \frac{\lambda}{\Theta_J}\int_{\mathbb{R}}\frac{\big(1+\theta(s,z)\big)^\eta - 1 - \eta\theta(s,z)}{\eta(\eta-1)}\nu(dz)\Big)\right\}ds \end{aligned} \tag{A.6}$$

where $d\left(Z_t\mathcal{U}_t^c\right)$ follows

$$
\begin{aligned}
dZ_t\mathcal{U}_t^c =& \partial_t\mathcal{U}\cdot Z_{t-}dt - \big(\mathcal{U}_{t-}^c + \partial_Z\mathcal{U}\cdot Z_{t-}\big)Z_{t-}\left(u_t dB^{\mathbb{B}} + \lambda\int_{\mathbb{R}}\theta(t,z)\nu(dz)dt\right)\\
&+ \partial_c\mathcal{U}\cdot Z_{t-}c_{t-}\left(\sigma^c dB^{\mathbb{B}} + \big(\mu_c + \lambda\int_{\mathbb{R}} z\nu(dz)\big)dt\right)\\
&+ \frac{1}{2}\big(\partial_{ZZ}^2\mathcal{U}\cdot Z_{t-} + 2\partial_Z\mathcal{U}\big)(u_t Z_{t-})^2 dt + \frac{1}{2}\partial_{cc}^2\mathcal{U}\cdot Z_{t-}(\sigma_c c_{t-})^2 dt - \big(\partial_c\mathcal{U} + \partial_{Zc}^2\mathcal{U}\cdot Z_{t-}\big)(u_t Z_{t-})(\sigma_c c_{t-})dt\\
&+ \int_{\mathbb{R}}\Big\{Z_{t-}(1+\theta(t,z)\cdot\mathcal{U}\big(t, c_{t-}(1-z), Z_{t-}(1+\theta(t,z))\big) - Z_{t-}\cdot\mathcal{U}\big(t, c_{t-}, Z_{t-}\big)\Big\}N(dt,dz).
\end{aligned}
$$

Since (A.5) is a martingale under optimal distortions, we must have the drift of (A.6) equals to zero under $\mathbb{B}$. Factoring out $e^{-\delta t}Z_{t-}$, we have following HJB equation,

$$
\begin{aligned}
0 = \inf_{u,\theta} & -\delta\mathcal{U}_t^c + \partial_t\mathcal{U} - \big(\mathcal{U}_{t-}^c + \partial_Z\mathcal{U}\cdot Z_{t-}\big)\lambda\int_{\mathbb{R}}\theta(t,z)\nu(dz) + \partial_c\mathcal{U}c_{t-}\big(\mu_c + \lambda\int_{\mathbb{R}} z\nu(dz)\big)\\
&+ \frac{1}{2}\big(\partial_{ZZ}^2\mathcal{U}\cdot Z_{t-} + 2\partial_Z\mathcal{U}\big)u_t^2 Z_{t-} + \frac{1}{2}\partial_{cc}^2\mathcal{U}(\sigma_c c_{t-})^2 - \big(\partial_c\mathcal{U} + \partial_{Zc}^2\mathcal{U}\cdot Z_{t-}\big)u_t(\sigma_c c_{t-})\\
&+ \delta U\left(c_t\right) + \Psi_{t-}Z_{t-}^{\eta-1}\Big(\frac{1}{2\Theta_D}u_t^2 + \frac{\lambda}{\Theta_J}\int_{\mathbb{R}}\frac{\big(1+\theta(t,z)\big)^{\eta} - 1 - \eta\theta(t,z)}{\eta(\eta-1)}\nu(dz)\Big)\\
&+ \lambda\int_{\mathbb{R}}\Big\{(1+\theta(t,z)\cdot\mathcal{U}\big(t, c_{t-}(1-z), Z_{t-}(1+\theta(t,z))\big) - \mathcal{U}\big(t, c_{t-}, Z_{t-}\big)\Big\}\nu(dz)
\end{aligned}
\tag{A.7}
$$

Collect the terms depends on u, θ receptively,

$$
\inf_{u}\frac{1}{2\Theta_D}\Psi_{t-}Z_{t-}^{\eta-1}u_t^2 + \frac{1}{2}\big(\partial_{ZZ}^2\mathcal{U}\cdot Z_{t-} + 2\partial_Z\mathcal{U}\big)Z_{t-}u_t^2 - \big(\partial_c\mathcal{U} + \partial_{Zc}^2\mathcal{U}\cdot Z_{t-}\big)(\sigma_c c_{t-})u_t \tag{A.8}
$$

$$
\begin{aligned}
&\inf_{\theta}\frac{\lambda}{\Theta_J}\Psi_{t-}Z_{t-}^{\eta-1}\int_{\mathbb{R}}\frac{\big(1+\theta(t,z)\big)^{\eta} - 1 - \eta\theta(t,z)}{\eta(\eta-1)}\nu(dz)\\
&\quad + \lambda\int_{\mathbb{R}}\Big\{(1+\theta(t,z))\mathcal{U}\big(t, c_{t-}(1-z), Z_{t-}(1+\theta(t,z))\big) - \theta(t,z)\big(\mathcal{U}_t^c + \partial_Z\mathcal{U}\cdot Z_{t-}\big)\Big\}\nu(dz)
\end{aligned}
\tag{A.9}
$$

Proposition 1 follows the same proof as Proposition 1 in Maenhout et al., 2021. For θ, the first-order condition yields

$$
\begin{aligned}
\frac{1 - \big(1+\theta(t,z)\big)^{\eta-1}}{\eta-1}\frac{1}{\Theta_J}\Psi_{t-}Z_{t-}^{\eta-1} =& \underbrace{\mathcal{U}\big(t, c_{t-}(1+\gamma(z)), Z_{t-}(1+\theta(t,z))\big) - \mathcal{U}_t^c}_{\equiv\Lambda_t}\\
&+ \underbrace{\partial_Z\mathcal{U}\big(t, c_{t-}(1-z), Z_{t-}(1+\theta(t,z))\big)\cdot(1+\theta(t,z))Z_{t-} - \partial_Z\mathcal{U}\cdot Z_{t-}}_{=Z_{t-}\partial_Z\Lambda_t\equiv F_t\Lambda_t}
\end{aligned}
\tag{A.10}
$$

where Λ describes the utility change and $\frac{\Lambda F}{Z}$ describes the change in $\partial_Z \mathcal{U}$, due to jumps in c. The optimal jump distortion is

$$\theta^*(t,z) = \left[1 - \frac{(\eta-1)\Theta_J \Lambda_t [1+F_t]}{\Psi_{t-}} Z_{t-}^{1-\eta}\right]^{\frac{1}{\eta-1}} - 1 \tag{A.11}$$

Since pessimism means that $\theta > 0$ and we assume random jump size $z > 0$. When z and $\theta(t,z)$ are infinitesimally small, Λ_t defined above is approximately $-\partial_c \mathcal{U} c_t z + \partial_Z \mathcal{U} Z_t \theta(t,z)$. Because $z > 0$ and $\theta > 0$, the previous expression resemble $-\Gamma_t$ of the diffusion distortion. When $\Gamma > 0$, we expect $\Lambda < 0$.

Consider $\eta < 1$ and a positive shock to Poisson process N. Because $\theta > 0$ when pessimism, Z in (A.3) increases after the shock and $Z^{1-\eta}$ increases as well. Then, holding $\Lambda[1+F] < 0$ unchanged and $\Psi_t > 0$, $[1 - \frac{(\eta-1)\Theta_J \Lambda_t [1+F_t]}{\Psi_{t-}} Z_{t-}^{1-\eta}]^{\frac{1}{\eta-1}}$ is larger than 1 and increases when $Z^{1-\eta}$ increases. As a result, $\theta^*(t,z)$ increases. Similar analysis can be done when $\eta > 1$.

When $\eta = 1$, the terms depending on θ in HJB equation (A.7) becomes

$$\begin{aligned}
&\inf_\theta \frac{\lambda}{\Theta_J} \Psi_{t-} \int_{\mathbb{R}} \Big[\big(1+\theta(t,z)\big)\log\big(1+\theta(t,z)\big) - \theta(t,z)\Big]\nu(dz) \\
&\quad + \lambda \int_{\mathbb{R}} \Big\{(1+\theta(t,z))\mathcal{U}\big(t, c_{t-}(1-z), Z_{t-}(1+\theta(t,z))\big) - \theta(t,z)\big(\mathcal{U}_t^c + \partial_Z \mathcal{U} \cdot Z_{t-}\big)\Big\}\nu(dz)
\end{aligned}$$

The first-order condition respect to θ from pervious equation,

$$-\frac{1}{\Theta_J}\Psi_{t-}\log\big(1+\theta(t,z)\big) = \Lambda_t[1+F_t]$$

gives the optimal θ for $\eta = 1$ in (18).

Suppose that $\int_{\mathbb{R}} \nu(dz) < \infty$. If θ is restricted to processes independent of z, then the first order condition of θ in (A.10) is transformed to

$$\frac{1-(1+\theta_t)^{\eta-1}}{\eta-1}\frac{1}{\Theta}\Psi_{t-} Z_{t-}^{\eta-1} \int_{\mathbb{R}} \nu(dz) = \int_{\mathbb{R}} (\Lambda_t + Z_{t-}\partial_Z \Lambda_t)\,\nu(dz)$$

where θ_t replaces $\theta(t,z)$ in the definitions of Λ_t and F_t is redefined as $F_t \mathbb{E}^{\nu(dz)}[\Lambda_t] \equiv Z_{t-}\mathbb{E}^{\nu(dz)}[\partial_Z \Lambda_t]$.

A.3 Proof of Proposition 3

Before proving Proposition 3, we first prepare the following result.

Lemma 2. *Given Z_1, the worest-case belief distortions in the second stage are given by*

$$u_1^* = \Theta_D Z_1^{1-\eta}\sigma\pi_1, \tag{A.12}$$

$$\theta_1^* = \left[1 - \frac{(\eta-1)}{1-\gamma}\Theta_J Z_1^{1-\eta}\mathbb{E}_\nu\left[(1-\pi_1 J)^{1-\gamma} - 1\right]\right]^{\frac{1}{\eta-1}} - 1, \tag{A.13}$$

where J is random variable with distribution ν. The optimal portfolio weight π_1^* satisfies

$$\mu - r - \sigma u_1^* + \lambda\bar{J} - \gamma\pi_1^*\sigma^2 - \lambda(1+\theta_1^*)\,\mathbb{E}_\nu\left[(1-\pi_1^* J)^{-\gamma} J\right] = 0. \tag{A.14}$$

The optimal consumption-wealth ratio $\tilde{c}_1^* = \delta^{\frac{1}{\gamma}} f(Z)^{-\frac{1}{\gamma}}$.

Proof. Recall from (23) and (24), the wealth dynamic under $\mathbb{U}$ follows

$$\frac{dW_t}{W_{t^-}} = \left[r + \pi_t(\mu - r) - \pi_t\sigma u_t - \pi_t\lambda\theta_t\bar{J} - \frac{c_t}{W_{t^-}}\right]dt + \pi_t\sigma dB_t^{\mathbb{U}} - \pi_t\big(J_t dN_t - \lambda(1+\theta_t)\bar{J}dt\big) \tag{A.15}$$

The value function in the second stage satisfies the following HJB equation:

$$\begin{aligned} 0 = -\delta V + \inf_{u_1,\theta_1}\max_{\pi,c} \Big\{ & \delta U(c) + (1-\gamma)VZ_1^{\eta-1}P(u_1,\theta_1) \\ & + \frac{\partial V}{\partial W}W\Big[r + \pi(\mu - r) - \pi\sigma u_1 - \pi\bar{J}\lambda\theta_1 - \frac{c_t}{W}\Big] + \frac{1}{2}\frac{\partial^2 V}{\partial W^2}W^2\pi^2\sigma^2 \\ & + (1+\theta_1)\lambda\Big\{V\,\mathbb{E}_\nu\left[(1-\pi J)^{1-\gamma} - 1\right] + \frac{\partial V}{\partial W}W\pi\bar{J}\Big\}\Big\}. \end{aligned} \tag{A.16}$$

The value function admits the decomposition $V(W,Z) = \frac{1}{1-\gamma}W^{1-\gamma}f(Z)$. Plugging this decomposition into (A.16), we obtain the following first-order condition for u_1, θ_1 and c_1:

$$\begin{aligned} u_1: &\quad \frac{1-\gamma}{\Theta_D}VZ_1^{\eta-1}u_1 - (1-\gamma)\pi\sigma V = 0 \quad \Rightarrow u_1^* = \Theta_D Z_1^{1-\eta}\sigma\pi. \\ \theta_1: &\quad \frac{1-\gamma}{\Theta_J}VZ^{\eta-1}\frac{(1+\theta_1)^{\eta-1} - 1}{(\eta-1)} + V\,\mathbb{E}_\nu\left[(1-\pi J)^{1-\gamma} - 1\right] = 0 \\ &\quad \Rightarrow \theta_1^* = \left[1 - \frac{(\eta-1)}{1-\gamma}\Theta_J Z_1^{1-\eta}\mathbb{E}_\nu\left[(1-\pi J)^{1-\gamma} - 1\right]\right]^{\frac{1}{\eta-1}} - 1. \\ c_1: &\quad \delta c^{-\gamma} - (1-\gamma)\frac{V}{W} = 0 \quad \Rightarrow c_1^* = \delta^{\frac{1}{\gamma}} f(Z)^{-\frac{1}{\gamma}}W_1 \end{aligned}$$

Collect and maximize the terms with π in (A.16),

$$\max_{\pi}\,(1-\gamma)V\left[(\mu-r)-\sigma u_1+\lambda\bar{J}\right]\pi-\frac{1}{2}\gamma(1-\gamma)V\pi^2\sigma^2+\lambda(1+\theta_1)V\,\mathbb{E}_\nu\left[(1-\pi J)^{1-\gamma}-1\right].$$

The first-order condition for π is given in (A.14).

Because $1-\gamma$ and V are of the same sign, meanwhile $\Theta_D,\Theta_J>0$, the second order conditions are satisfied as well:

$$\begin{aligned}
u_1:&\quad \frac{1-\gamma}{\Theta_D}VZ_1^{\eta-1}\geq 0,\\
\theta_1:&\quad \frac{1-\gamma}{\Theta_J}V\cdot Z_1^{\eta-1}(1+\theta_1)^{\eta-2}\geq 0,\\
c_1:&\quad -\delta c^{-\gamma-1}<0\\
\pi:&\quad -\gamma(1-\gamma)V\sigma^2-\gamma(1-\gamma)\lambda(1+\theta_1)V\,\mathbb{E}_\nu\left[(1-\pi J)^{-\gamma-1}J^2\right]\leq 0.
\end{aligned}$$

Finally, the optimal belief distortions and consumption-investment strategies in the second stage are the constants identified by above first-order conditions. □

The π is a admissible strategy, where $\pi_t^*J_{N_t}\leq 1$.

Proof of Proposition 3. Suppose that π_1^* decreases as $Z_1^{\eta-1}$ increases, we will derive a contradict.

First, as $Z_1^{\eta-1}$ increases and π_1^* decreases, (A.12) implies that u_1^* decreases. Second, the admissibility of π^* ensures that $\pi_1^*J<1$. Therefore, $-\frac{1}{1-\gamma}\mathbb{E}_\nu\left[(1-\pi_1^*J)^{1-\gamma}-1\right]$ decreases as π_1^* decreases. As $Z^{\eta-1}$ increases, (A.13) implies that θ_1^* decreases. Third, the first-order condition for π_1 in (A.14) can be decomposed into three parts:

$$\begin{aligned}
&\text{Part 1}:(\mu-r)-\sigma u_1^*+\lambda\bar{J},\\
&\text{Part 2}:-\gamma\pi_1^*\sigma^2,\\
&\text{Part 3}:-\lambda(1+\theta_1^*)\,\mathbb{E}_\nu\left[(1-\pi_1^*J)^{-\gamma}J\right].
\end{aligned}$$

Part 1 increases because u_1^* decreases. Part 2 increases because we assume that π_1^* decreases. For Part 3,

$$\partial_{\pi_1^*}\mathbb{E}_\nu\left[(1-\pi_1^*J)^{-\gamma}J\right]>0,$$

for π_1^* satisfying $\pi_1^*J<1$. Then $\mathbb{E}_\nu\left[(1-\pi_1^*J)^{-\gamma}J\right]$ is positive and decreases as π_1^* decreases. We have seen from the previous argument that $1+\theta_1^*$ is positive and decreases. Therefore, Part 3 increases as $Z_1^{\eta-1}$ increases and π_1^* decreases. In summary, we have shown that all parts on the left-hand side of the first-order condition (A.14) increase as π_1^* and $Z_1^{\eta-1}$ decrease. This contradicts with the first order condition (A.14). Therefore, π_1^* must increases when $Z_1^{\eta-1}$ increases. $\square$

A.4 Proof of Proposition 4

Recall Z^D and Z^J from (1) and (2). They satisfy

$$\begin{aligned}
Z_t^D &= \exp\Big(-\int_0^t u_s dB_s^{\mathbb{B}} - \frac{1}{2}\int_0^t |u_s|^2 ds \Big), \\
Z_t^J &= \exp\Big\{ \int_0^t \ln(1+\theta_s) d\widetilde{N}_s^{\mathbb{B}} + \int_0^t \lambda\big(\ln(1+\theta_s) - \theta_s\big) ds \Big\}.
\end{aligned}$$

Therefore, x, defined in (31), follows the dynamics

$$\begin{aligned}
dx_t &= u_t dB_t^{\mathbb{B}} - \ln\big(1+\theta_t\big) d\widetilde{N}_t^{\mathbb{B}} + \left(\frac{1}{2}u_t^2 - \lambda\Big(\ln(1+\theta_t) - \theta_t\Big)\right) dt \\
&= u_t dB_t^{\mathbb{U}} - \ln\big(1+\theta_t\big) \widetilde{N}_t^{\mathbb{U}} - \left(\frac{1}{2}u_t^2 + \lambda\Big((1+\theta_t)\ln(1+\theta_t) - \theta_t\Big)\right) dt,
\end{aligned}$$

where $\widetilde{N}_t^{\mathbb{B}} = N_t - \lambda t$ and $\widetilde{N}_t^{\mathbb{U}} = N_t - \lambda(1+\theta_t)t$ are compensated Poisson process under $\mathbb{B}$ and $\mathbb{U}$, respectively.

We will first derive the HJB equation that the function f, defined via (35), satisfies. To this end, the wealth dynamics follows

$$\frac{dW_t}{W_{t^-}} = \Big[r + \pi_t(\mu - r) - \pi_t\sigma u_t + \pi_t\lambda\bar{J} - \tilde{c}_t\Big]dt + \pi_t\sigma dB_t^{\mathbb{U}} - \pi_t J_t dN_t, \tag{A.17}$$

where $\tilde{c}_t = \frac{c_t}{W_{t^-}}$ is the consumption-wealth ratio. Itô formula for jump diffusions implies that $W_t^{1-\gamma}$ and $e^{f(t,x_t)}$ follow the dynamics

$$\begin{aligned}
\frac{dW_t^{1-\gamma}}{W_{t^-}^{1-\gamma}} &= (1-\gamma)\left(r + \pi_t(\mu-r) - \pi_t\sigma u_t + \pi_t\lambda\bar{J} - \tilde{c}_t - \frac{1}{2}\gamma\pi_t^2\sigma^2\right) dt + (1-\gamma)\pi_t\sigma dB_t^{\mathbb{U}} \\
&\quad + \left((1-\pi_t J)^{1-\gamma} - 1\right) dN_t \qquad \text{(A.18)} \\
\frac{de^{f(t,x_t)}}{e^{f\left(t,x_{t^-}\right)}} &= \partial_t f dt + \partial_x f\left(u_t dB_t^{\mathbb{U}} - \left(\frac{1}{2}u_t^2 - \lambda\theta_t\right) dt\right) + \frac{1}{2}\left(\partial_x f^2 + \partial_{xx} f\right) u_t^2 dt \\
&\quad + \left(e^{f\left(t^-, x_{t^-} - \ln(1+\theta_t)\right) - f(t^-, x_{t^-})} - 1\right) dN_t \qquad \text{(A.19)}
\end{aligned}$$

It then follows from the martingale principle that

$$e^{-\delta t}V_t+\int_0^t e^{-\delta s}\left\{\delta U\left(c_s\right)+(1-\gamma)V_{s-}e^{(1-\eta)x_{t-}}\left(\frac{1}{2\Theta_D}u_s^2+\frac{\lambda}{\Theta_J}\frac{\left(1+\theta_s\right)^\eta-1-\eta\theta_s}{\eta(\eta-1)}\right)\right\}ds \tag{A.20}$$

is $\mathbb{U}$-martingale under optimal (π^*, c^*) and (u^*, θ^*), a $\mathbb{U}$-supermartingale when an arbitrary (π, c) and the optimal (u^*, θ^*) are employed, and a $\mathbb{U}$-submartingale when the optimal (π^*, c^*) and an arbitrary (u, θ) are employed. Applying Itô formula and using the decomposition (35), we obtain the dynamics of the process in (A.20) as

$$\begin{aligned}&-\delta e^{-\delta t}\frac{W_t^{1-\gamma}}{1-\gamma}e^{f(t,x_t)}dt+\frac{e^{-\delta t}}{1-\gamma}d\left(W_t^{1-\gamma}e^{f(t,x_t)}\right)\\&\quad+e^{-\delta t}\left\{\delta U\left(c_t\right)+(1-\gamma)\frac{W_{t^-}^{1-\gamma}}{1-\gamma}e^{f\left(t,x_{t^-}\right)}e^{(1-\eta)x_{t^-}}\left(\frac{1}{2\Theta_D}u_t^2+\frac{\lambda}{\Theta_J}\frac{\left(1+\theta_t\right)^\eta-1-\eta\theta_t}{\eta(\eta-1)}\right)\right\}.\end{aligned} \tag{A.21}$$

Meanwhile $d\left(W_t^{1-\gamma}e^{f(t,x_t)}\right)$ follows

$$\begin{aligned}\frac{d\left(W_t^{1-\gamma}e^{f(t,x_t)}\right)}{W_{t^-}^{1-\gamma}e^{f\left(t^-,x_{t^-}\right)}}=&(1-\gamma)\left[\left(r+\pi_t\left(\mu-r-\sigma u_t+\lambda\bar{J}\right)-\tilde{c}_t-\frac{1}{2}\gamma\pi_t^2\sigma^2+\partial_x f\pi_t\sigma u_t\right)dt+\sigma dB_t^{\mathbb{U}}\right]\\&+\left(\partial_t f-\partial_x f\left(\frac{1}{2}u_t^2-\lambda\theta_t\right)+\frac{1}{2}\left(\left(\partial_x f\right)^2+\partial_{xx}f\right)u_t^2\right)dt+\partial_x f u_t dB_t^{\mathbb{U}}\\&+\left[\left(1-\pi_t J_t\right)^{1-\gamma}e^{f\left(t^-,x_{t^-}-\ln(1+\theta_t)\right)-f\left(t^-,x_{t^-}\right)}-1\right]dN_t.\end{aligned} \tag{A.22}$$

Combining the previous two dynamics and utilizing the martingale principle, we obtain the following HJB equation:

$$\begin{aligned}0=\sup_{\pi,\tilde{c}}\inf_{\theta,u}\Big\{&-\frac{\delta}{1-\gamma}+r+\pi\left(\mu-r-\sigma u+\lambda\bar{J}\right)-\tilde{c}+\partial_x f\pi\sigma u-\frac{1}{2}\gamma\pi^2\sigma^2\\&+\frac{1}{1-\gamma}\left[\partial_t f-\partial_x f\left(\frac{1}{2}u^2-\lambda\theta\right)+\frac{1}{2}\left(\left(\partial_x f\right)^2+\partial_{xx}f\right)u^2\right]\\&+\frac{\lambda(1+\theta)}{1-\gamma}\left\{\mathbb{E}^{\mathbb{B}}\left[(1-\pi J)^{1-\gamma}\right]e^{f(t,x-\ln(1+\theta))-f(t,x)}-1\right\}\\&+\frac{\delta}{1-\gamma}\tilde{c}^{1-\gamma}e^{-f(t,x)}+e^{(1-\eta)x}\left(\frac{1}{2\Theta_D}u^2+\frac{\lambda}{\Theta_J}\frac{\left(1+\theta\right)^\eta-1-\eta\theta}{\eta(\eta-1)}\right)\Big\}.\end{aligned} \tag{A.23}$$

The optimal belief distortion (u^*, θ^*), portfolio weight π^* and consumption-wealth ratio $\tilde{c}^*$ are obtained by first-order conditions, when the following second order conditions are satisfied:

1. $\gamma^{\text{eff}}(t,x)\sigma^2$ is positive definite,

2. $(1-\gamma)\big[(\partial_x f)^2 + \partial_{xx} f - \partial_x f + \frac{1-\gamma}{\Theta} e^{(1-\eta)x}\big] > 0$,

3. $|\partial_x f|$ cannot be too large, i.e., cannot be $<< 0$ or $>> 1$.

When x is $\underline{x}$ or $\bar{x}$, the boundary conditions of f are specified by the value f^{fr}, which is independent of x, and it satisfies the following ODE, obtained by setting $\partial_x f = 0$, $\partial_{xx} f = 0$, and removing terms related to jumps in x from the HJB equation (A.23):

$$0 = \sup_{\pi,\tilde{c}} \inf_{\theta,u} \Big\{ \frac{\partial_t f}{1-\gamma} - \frac{\delta}{1-\gamma} + r + \pi\left(\mu - r - \sigma u + \lambda \bar{J}\right) - \tilde{c} - \frac{1}{2}\gamma\pi^2\sigma^2 + \frac{\lambda(1+\theta)}{1-\gamma}\left\{\mathbb{E}_\nu\left[(1-\pi J)^{1-\gamma}\right] - 1\right\} + \frac{\delta}{1-\gamma}\tilde{c}^{1-\gamma} e^{-f} + e^{(1-\eta)x}\Big(\frac{1}{2\Theta_D}u^2 + \frac{\lambda}{\Theta_J}\frac{\left(1+\theta\right)^\eta - 1 - \eta\theta}{\eta(\eta-1)}\Big)\Big\}.$$

The first-order conditions give the optimal values for (u,θ,π,c) at the boundary, denoted as $(u^{\text{fr}}, \theta^{\text{fr}}, \pi^{\text{fr}}, c^{\text{fr}})$. The second order conditions are satisfied if $\theta^{\text{fr}} > 0$ and π is admissible.

B Additional Plots and Table

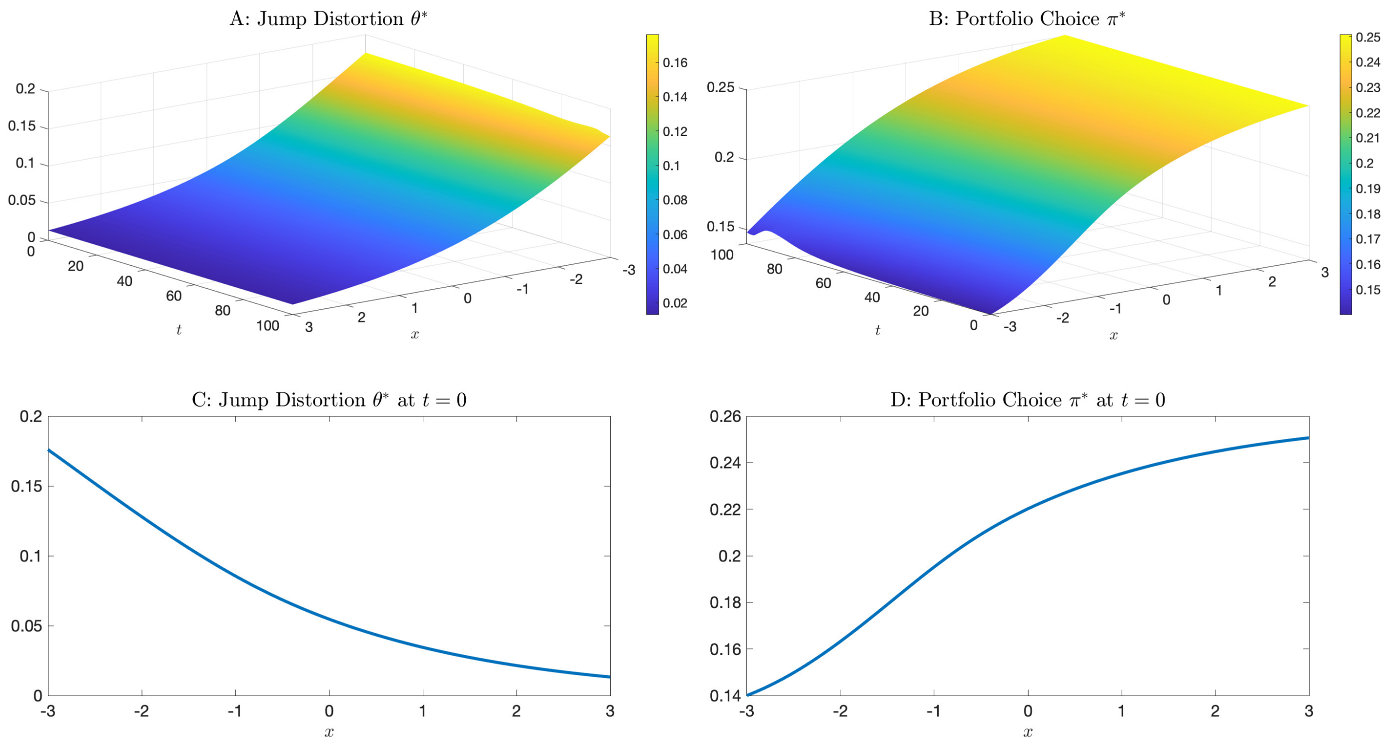

Figure B.1: Optimal distortion and portfolio with only jump distortion

This figure plots optimal distortions and portfolios only with jump penalty. The time horizon is T=100 years, and the parameters used are summarized in Table 2.

By forcing u equal to zero, we shut down the diffusion distortion and focus on the impact of jump distortion. In this case, the impact of jump belief distortion on the portfolio is isolated. From Figure B.1, jump distortion and portfolio position will change in the opposite direction over time. With a penalty only for jump, jump distortion slightly changes over time, mainly changing with the state variable. This is because the investors are forward-looking. Unless they expect a massive jump with high intensity in the future (i.e., the sentiment state changes), they will not change their belief of market crashes dramatically.

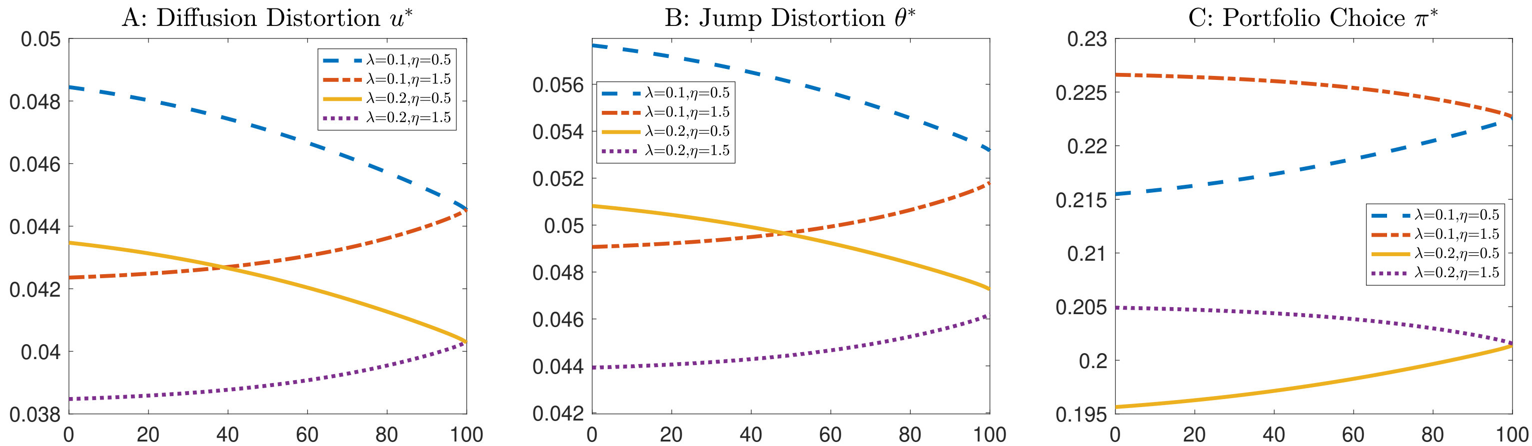

Figure B.2: Impact of jump intensity

This figure plots optimal distortions and portfolios at $x = 0$. The time horizon is T=100 years, and the parameters used are summarized in Table 2.

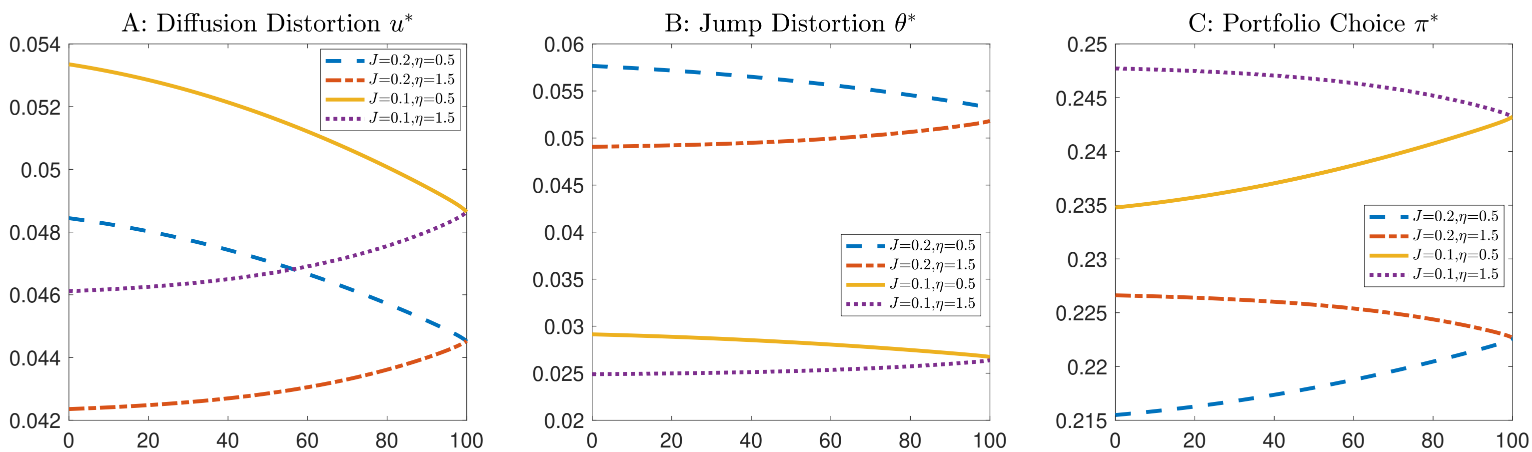

Figure B.3: Impact of jump size

This figure plots optimal distortions and portfolios at $x = 0$. The time horizon is T=100 years, and the parameters used are summarized in Table 2.

Additionally, we can modify the jump intensity or size to examine the impact of jumps. An increase in jump intensity or size results in greater jump risk and a reduction in subjective expected return. Consequently, the agent holds fewer positions in the risky asset, as illustrated in Panel C of both Figure B.2 and Figure B.3. It is also evident that jump risk influences not only jump belief but also diffusion belief over time. In Figure B.2, the agent assigns a larger penalty to model uncertainty as jump intensity increases. In Figure B.3, when the jump size is larger, the agent under-reacts to diffusion risk and over-reacts to jump risk. Both cases conform to the same arguments presented in the state dependence section.

Table B.1: Sensitivity Estimation

	Model Estimation
State-dependent	11.846
	(0.002)
State-independent ($\eta = 1$)	10.753
Jump-Diffusion	15.844
Merton	19.580
Giglio et al., 2021	1.164
	(0.061)

The regression for state-dependent case is based on the simulated data in the fifth year, with T = 30 years. The parameters used for all four cases are summarized in Table 3.

Giglio et al., 2021 finds a relatively small sensitivity of equity share in investors' holding to shocks in expectations from survey data. Using simulated data based on our model, we estimate the sensitivity β of equity shares to expected excess return,

$$\text{EquityShare}_{i,t} = \alpha + \beta(E_i[R] - R_f) + \epsilon_i \tag{B.1}$$

where intercept α should be zero under the rational expectation framework. To compare with the cross-sectional result in Giglio et al., 2021), we take the simulated panel data at fifth year to estimated the sensitivity, allowing agents' sentiments generating diversified dynamics. From Table B.1, one can observe that increase in the subjective expected return will bring less impact on the portfolio in our model, compared with jump-diffusion or Merton model, while being larger than Giglio et al., 2021's 1.164, for samples with expected returns around 5%. In Merton model, β corresponds to $\frac{1}{\gamma\sigma^2}$, where stock volatility σ is fixed at 15.98% in our simulation setting. In other model settings, investors implicitly have higher risk aversion levels, which causes less sensitive demand. Two main factors drive risk aversion to increase: first is jump risk, which can be observed by comparing β in jump-diffusion and the Merton model. Even though the jump is compensated, an additional risk source will make investors more risk-averse. A novel factor in this paper is model uncertainty. With robustness concerned, a pessimistic investor distorts her belief away from baseline belief, leading to a higher effective risk aversion level γ^{eff}. Besides, investors with state-independent penalty tend to have smaller risk aversion than state-dependent ones, as the simulated sentiment variable x has positive drift and state-dependent investors tend to be less pessimistic.

C Special case $\eta = 1$

When $\eta = 1$, the value function is independent of the state varible x. Consider the jump intensity uncertainty only, the HJB equation becomes

$$0 = \sup_{\pi,\tilde{c}} \inf_{\theta,u} \frac{\partial_t f}{1-\gamma} + \frac{\delta}{1-\gamma}\tilde{c}^{1-\gamma}e^{-f} + r - \tilde{c}_t - \frac{\delta}{1-\gamma} + \left(\frac{1}{2\Theta_D}u_t^2 + \frac{\lambda}{\Theta_J}\left((1+\theta_t)\log\left(1+\theta_t\right) - \theta_t\right)\right)$$
$$+ \pi_t\left(\mu - r - \sigma u_t + \lambda \bar{J}\right) - \frac{\gamma}{2}\pi_t^2\sigma^2 + \frac{\lambda\left(1+\theta_t\right)}{1-\gamma}\mathbb{E}^{\mathbb{B}}\left[\left(1-\pi_t J_t\right)^{1-\gamma} - 1\right] \tag{C.1}$$

F.O.C. gives

$$u_t : u_t = \Theta_D \sigma \pi_t \tag{C.2}$$

$$\theta_t : 0 = \frac{1}{\Theta_J}\log(1+\theta_t) + \frac{1}{1-\gamma}\left\{\mathbb{E}^{\mathbb{B}}\left[\left(1-\pi_t J_t\right)^{1-\gamma}\right] - 1\right\} \tag{C.3}$$

$$\pi_t : \mu + \lambda\bar{J} - r = (\gamma + \Theta_D)\pi_t\sigma^2 + \lambda\left(1+\theta_t\right)\mathbb{E}^{\mathbb{B}}\left[\left(1-\pi_t J_t\right)^{-\gamma} J_t\right] \tag{C.4}$$

As we see from the numerical result in Figure 6, θ_t will not converge to $\eta = 1$ case when $T = 100$. To see the dependency between θ and η, we can assume after τ, state variable x doesn't change and stays at 0. For $\eta \neq 1$, the first-order condition (38) becomes

$$0 = \frac{1}{\Theta_J}\frac{\left(1+\theta_\tau\right)^{\eta-1} - 1}{\eta - 1} + \frac{1}{1-\gamma}\left\{\mathbb{E}^{\mathbb{B}}\left[\left(1-\pi_\tau J_\tau\right)^{1-\gamma}\right] - 1\right\} \tag{C.5}$$

If π, u converge to the same values as $\eta = 1$, by first-order conditions for different η, we have

$$\frac{\left(1+\theta_\tau\right)^{\eta-1} - 1}{\eta - 1} = \log\left(1+\theta_\tau\right) \tag{C.6}$$

where optimal θ_τ depends on the value of η.

D Numerical algorithm

In this section, we present numerical framework to solve the HJB equations in Proposition 4 with the finite difference method. Here, the jump size J is treated as a random variable. In implications part, J is specified as a constant level.

First, discretize x from $x_{\min}$ to $x_{\max}$ with equal spaced grid size Δx, and discretize t from 0 to T with equal spaced grid size Δt. The function f associated with HJB (36) is evaluated on grid point $(m\Delta x, n\Delta t)$. Take $x(m) = x_{\min} + m\Delta x$ and $t(n) = n\Delta t$ for $m = 1, \dots, M$ and $n = 0, \dots, N$, denote $f(m,n) = f(x(m), t(n))$.

We define

$$I_t = \mathbb{E}^{\mathbb{B}}\left[(1-\pi_t J_t)^{1-\gamma}\right] = \int_{\mathbb{R}} (1-\pi_t J_t)^{1-\gamma}\,\nu(dJ) \tag{D.1}$$

$$I_t' = \mathbb{E}^{\mathbb{B}}\left[(1-\pi_t J_t)^{-\gamma} J_t\right] = \int_{\mathbb{R}} (1-\pi_t J_t)^{-\gamma} J_t \nu(dJ) \tag{D.2}$$

and approximate $f\left(x(m) - \ln\left(1+\theta(m,n)\right), t(n)\right) - f\left(m,n\right)$ by

$$\Delta f_{mid}(m, n+1) = f(m-k, n+1)w + f(m-k-1, n+1)(1-w) - f(m,n) \tag{D.3}$$

where $\omega = \frac{(k+1)\Delta x - \ln(1+\theta(m,n))}{\Delta x}$ if $\ln(1+\theta(m,n)) \in [k\Delta x, (k+1)\Delta x]$.

Discretization of HJB (36) is

$$\begin{aligned}
0 =& \frac{f(m,n+1)-f(m,n)}{\Delta t} + \frac{1}{2}u^2(m,n)\Big[\frac{f(m+1,n)+f(m-1,n)-2f(m,n)}{(\Delta x)^2} \\
&+ \left(\partial_x f(m,n+1) - 1\right)\frac{f(m+1,n)-f(m-1,n)}{2\Delta x}\Big] \\
&+ \frac{f(m+1,n)-f(m-1,n)}{2\Delta x}\Big[(1-\gamma)\pi(m,n)\sigma u(m,n) + \lambda\theta(m,n)\Big] \\
&+ \gamma\delta^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma}f(m,n+1)} - \delta + (1-\gamma)e^{(1-\eta)x(m)}\left(\frac{1}{2\Theta_D}u^2(m,n) + \frac{\lambda}{\Theta_J}\frac{(1+\theta(m,n))^\eta - 1 - \eta\theta(m,n)}{\eta(\eta-1)}\right) \\
&+ (1+\theta(m,n))\lambda\left[e^{\Delta f_{mid}(m,n+1)}\cdot I(m,n) - 1\right] \\
&+ (1-\gamma)\left[r + \pi(m,n)\left(\mu - r - \sigma u(m,n) + \lambda\bar{J}\right) - \frac{1}{2}\gamma\sigma^2\pi(m,n)^2\right]
\end{aligned}$$

Rearrange the above equation, we obtain

$$\left(1+\frac{\Delta^t}{(\Delta x)^2}u^2(m,n)\right)f(m,n)=$$
$$\left\{\frac{1}{2}u^2(m,n)\Big[\frac{\Delta t}{(\Delta x)^2}+\frac{\Delta t}{2\Delta x}(\partial_x f(m,n+1)-1)\Big]+\frac{\Delta t}{2\Delta x}\Big[(1-\gamma)\pi(m,n)\sigma u(m,n)+\lambda\theta(m,n)\Big]\right\}f(m+1,n)$$
$$+\left\{\frac{1}{2}u^2(m,n)\Big[\frac{\Delta t}{(\Delta x)^2}-\frac{\Delta t}{2\Delta x}(\partial_x f(m,n+1)-1)\Big]-\frac{\Delta t}{2\Delta x}\Big[(1-\gamma)\pi(m,n)\sigma u(m,n)+\lambda\theta(m,n)\Big]\right\}f(m-1,n)$$
$$+f(m,n+1)+\Delta t\Bigg\{\gamma\delta^{\frac{1}{\gamma}}e^{-\frac{1}{\gamma}f(m,n+1)}-\delta+(1+\theta(m,n))\lambda\left[e^{\Delta f_{mid}(m,n+1)}\cdot I(m,n)-1\right]$$
$$+(1-\gamma)\Big[r+\pi(m,n)\left(\mu-r-\sigma u(m,n)+\lambda\bar{J}\right)-\frac{1}{2}\gamma\sigma^2\pi(m,n)^2\Big]$$
$$+(1-\gamma)e^{(1-\eta)x(m)}\left(\frac{1}{2\Theta_D}u^2(m,n)+\frac{\lambda}{\Theta_J}\frac{(1+\theta(m,n))^\eta-1-\eta\theta(m,n)}{\eta(\eta-1)}\right)\Bigg\}$$

where $\theta(m,n),\pi(m,n),u(m,n)$ are calculated via F.O.Cs (37), (38) and (39) at time $n+1$. In code, $(\theta(m,n),\pi(m,n))$ are solved by minimizing an objective function, refer to (D.1). The terminal condition is $f(m,N)=\log\varepsilon$.

At the boundary, we deiscretization of ODE (41) at $m=0$ or M,

$$0=\frac{f(m,n+1)-f(m,n)}{\Delta t}+\gamma\delta^{\frac{1}{\gamma}}e^{-\frac{1}{\gamma}f(m,n)}+(1-\gamma)\left\{r+\pi^{\text{fr}}\left(\mu-r-\sigma u^{\text{fr}}+\lambda\bar{J}\right)\right\}-\frac{1}{2}\gamma(1-\gamma)\sigma^2(\pi^{\text{fr}})^2$$
$$+(1-\gamma)e^{(1-\eta)x(m)}\left(\frac{1}{2\Theta_D}(u^{\text{fr}})^2+\frac{\lambda}{\Theta_J}\frac{\left(1+\theta^{\text{fr}}\right)^\eta-1-\eta\theta^{\text{fr}}}{\eta(\eta-1)}\right)-\delta+\lambda(1+\theta^{\text{fr}})\left[I(m,n)-1\right]$$

Rearrange the above equation, we obtain

$$f(m,n)=f(m,n+1)+\Delta t\Bigg\{-\delta+(1-\gamma)\left[r+\pi^{\text{fr}}\left(\mu-r-\sigma u^{\text{fr}}+\lambda\bar{J}\right)-\frac{1}{2}\gamma\sigma^2\left(\pi^{\text{fr}}\right)^2\right]$$
$$+(1-\gamma)e^{(1-\eta)x(m)}\left(\frac{1}{2\Theta_D}\left(u^{\text{fr}}\right)^2+\frac{\lambda}{\Theta_J}\frac{(1+\theta^{\text{fr}})^\eta-1-\eta\theta^{\text{fr}}}{\eta(\eta-1)}\right)$$
$$+\gamma\delta^{\frac{1}{\gamma}}e^{-\frac{1}{\gamma}f(m,n)}+\lambda\left(1+\theta^{\text{fr}}\right)\left[I(m,n)-1\right]\Bigg\}$$

Solve $\theta^{\text{fr}},\pi^{\text{fr}},u^{\text{fr}}$ by F.O.Cs with the root finding algorithm similar to Table D.1.

Table D.1

Algorithm for solving First-order conditions

1. Given initial $\pi^{\text{old}}(m,n)$ and $\theta^{\text{old}}(m,n)$, calculate:
   $I(m,n) = \sum_{k=0}^{k-1} (1 - \pi(m,n)J_k)\, \nu\,(J_k)\, \Delta J, \quad I'(m,n) = \sum_{k=0}^{k-1} (1 - \pi\,(m,n)\, J_k)\, J_k \nu\,(J_k)\, \Delta J$
   and $\Delta f_{\text{mid}}(m, n+1)$ in (D.3).

2. Update $\pi(m,n)$ and $\theta(m,n)$ with the values that minimize sum of squares of first-order conditions (39) and (38), where $u(m,n)$ substituted by $\pi(m,n)$ using (37).

3. Update $I(m,n)$, $I'(m,n)$, and $\Delta f_{\text{mid}}(m, n+1)$ with $\pi^{\text{new}}(m,n)$ and $\theta^{\text{new}}(m,n)$.

4. Return to the second step until convergence or max iteration is met.

5. Calculate $u(m,n)$ via the first-order condition (37).